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# Three-Dimensional Analysis of Acoustic Scattering from a Coated Cylindrical Shell

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A theoretical model was developed to evaluate the farfield pressure scattered from a coated cylindrical shell by a plane acoustic wave incident at an angle to the cylindrical axis (a three-dimensional problem). The model is a two-layer-structure comprising an outer layer of microvoided elastomer (coating) that is perfectly bonded to a cylindrical shell. The analysis of this model is based on the theory of elasticity, acoustic wave equations, and pertinent boundary conditions. The major results presented here are calculated directivity patterns using three-dimensional analysis.

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## LIST OF SYMBOLS

| | |
|---|---|
| $B$ | Complex bulk modulus |
| $c_d$ | Complex dilatational (compressional) wave speed |
| $c_{d1}$ | Complex dilatational wave speed in the coating (layer 1) |
| $c_{d2}$ | Complex dilatational wave speed in the cylindrical shell (layer 2) |
| $c_{d01}$ | Real part of complex dilatational wave speed in the coating |
| $c_{d02}$ | Real part of complex dilatational wave speed in the cylindrical shell |
| $c_s$ | Complex shear (transverse) wave speed |
| $c_{s1}$ | Complex shear wave speed in the coating |
| $c_{s2}$ | Complex shear wave speed in the cylindrical shell |
| $c_{s01}$ | Real part of complex shear wave speed in the coating |
| $c_{s02}$ | Real part of complex shear wave speed in the cylindrical shell |
| $c_0$ | Sound speed in water |
| $c_3$ | Sound speed in air |
| $f$ | Frequency in Hertz |
| $h_1$ | Coating thickness |
| $h_2$ | Shell thickness |
| $k_d = \omega / c_d$ | Dilatational wave number |
| $k_s = \omega / c_s$ | Shear wave number |
| $k_z$ | Wave number in axial direction |
| $k_0 = \omega / c_0$ | Acoustic wave number in water |
| $k_3 = \omega / c_3$ | Acoustic wave number in air |
| $p_i$ | Incident pressure field (wave) in water |
| $P_i$ | Complex amplitude of $p_i$ |
| $P_{af}$ | Amplitude factor |
| $p_s$ | Scattered pressure field (wave) in water |
| $p_0$ | Total pressure field (wave) in water |
| $p_3$ | Transmitted pressure field (wave) in air |
| $P_3$ | Complex amplitude of $p_3$ |
| $t$ | Time |
| $u_r$ | Radial displacement |
| $u_\theta$ | Circumferential displacement |
| $u_z$ | Axial displacement |
| $x, y, z, r, \theta$ | Spatial coordinates |

## LIST OF SYMBOLS (Cont'd)

| | |
|---|---|
| $\zeta_{d1}$ | Loss factor associated with dilatational wave in the coating |
| $\zeta_{d2}$ | Loss factor associated with dilatational wave in the shell |
| $\zeta_{s1}$ | Loss factor associated with shear wave in the coating |
| $\zeta_{s2}$ | Loss factor associated with shear wave in the shell |
| $\varphi$ | Angle of incidence measured relative to the normal (radial) direction |
| $\lambda, \mu$ | Lamé constants |
| $\rho$ | Material density |
| $\rho_0$ | Density of water |
| $\rho_1$ | Material density of coating |
| $\rho_2$ | Material density of shell |
| $\rho_3$ | Density of air |
| $\tau_{rr}$ | Normal stress in the radial direction |
| $\tau_{r\theta}$ | Shear stress in the circumferential direction |
| $\tau_{rz}$ | Shear stress in the axial direction |
| $\phi$ | Scalar potential |
| $\psi$ | Vector potential |
| $\psi_r$ | Radial component of $\psi$ |
| $\psi_\theta$ | Circumferential component of $\psi$ |
| $\psi_z$ | Axial component of $\psi$ |
| $\omega = 2\pi f$ | Angular frequency in radians/s |

# THREE-DIMENSIONAL ANALYSIS OF ACOUSTIC SCATTTERING FROM A COATED CYLINDRICAL SHELL

## INTRODUCTION

In this study, a theoretical model is developed for evaluating the farfield pressure scattered from an infinitely long, coated cylindrical shell by a plane acoustic wave incident at an angle with the normal to the cylindrical shell axis (a three-dimensional problem). The model is a two-layer-structure comprising an outer layer of microvoided elastomer (coating) that is perfectly bonded to a cylindrical shell. The coating consists of an acoustically soft material that is designed for reducing flexural wave noise.

## BACKGROUND

The acoustic reflection from a solid cylinder in water is described by Morse[1] and Bowen et al.[2] The scattering of sound by hollow elastic cylinders in water has been investigated by Gaunaurd[3] and Flax and Neubauer.[4] In these cases, the incident sound wave was taken as an infinite plane wave normally incident to the cylindrical axis (a two-dimensional problem). This wave excites the radial and circumferential modes. However, a plane wave incident at an angle with the normal to the cylindrical axis excites modes of vibration of the cylindrical shell having radial, circumferential, and axial dependence (a three-dimensional problem).

## SPECIFIC OBJECTIVE

This report describes the three-dimensional analysis for the two-layer coated cylindrical shell. An earlier two-dimensional problem[5] has been extended to the three-dimensional case to investigate the scattering of the plane acoustic wave by the coated cylindrical shell. Ko and Sandman[6] presented a portion of the two-dimensional results. The formulation of the problem is based on the theory of elasticity, acoustic wave equations, and pertinent boundary conditions.

The major accomplishment of this research is the numerical calculation of directivity patterns for the coated cylindrical shell using the three-dimensional analysis.

## THEORETICAL ANALYSIS OF ELASTIC WAVES IN CYLINDRICAL COORDINATES

The geometry of the theoretical model in this study is depicted in figure 1. The outer surface of the composite structure, which consists of the coating and the cylindrical shell, is in contact with water. The core (cavity) of the structure contains air.

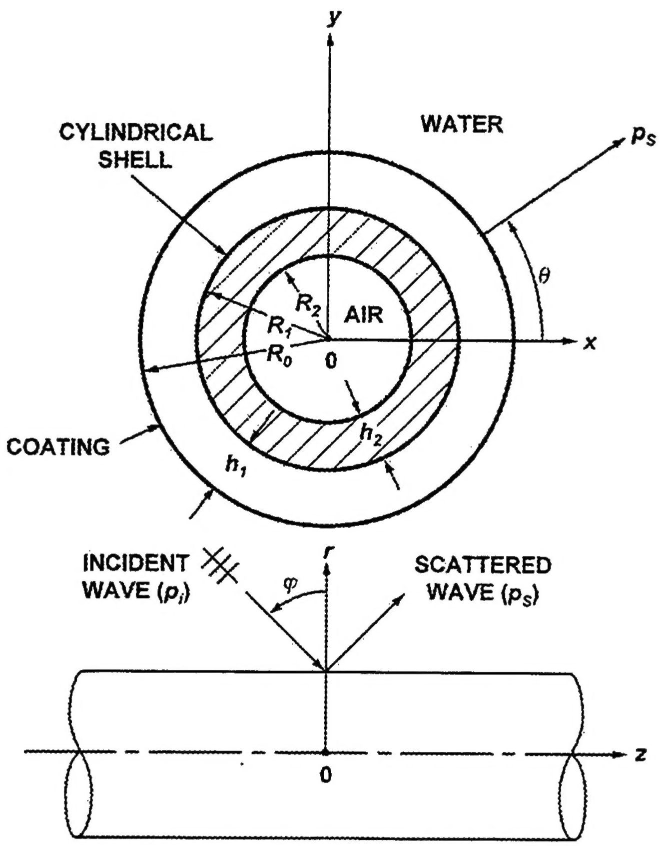


*Figure 1. Geometry of Theoretical Model*

It is essential to study waves propagating in elastic media. In this section, equations necessary for the formulation of the present problem are developed. The vector differential equation that governs the small elastic motion in the elastic medium is written as[7,8]

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{1}$$

where $\lambda$ and $\mu$ are the Lamé constants, $\boldsymbol{u}$ is the displacement vector, $\rho$ is the material density, $\nabla$ is the gradient operator, $t$ is the time, and $\nabla^2$ is the Laplacian operator. The solutions of equation (1) are a combination of a vector potential $\psi$ and a scalar potential function $\phi$, so that

$$\boldsymbol{u} = \nabla\phi + (\nabla\times\psi), \tag{2}$$

which is true provided that $\phi$ and $\psi$ are the solutions of the elastic wave equations

$$\nabla^2\phi = \frac{1}{c_d^2}\frac{\partial^2\phi}{\partial t^2}, \tag{3}$$

$$\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}, \tag{4}$$

and

$$\nabla\cdot\psi = F, \tag{5}$$

where $F$ is an arbitrary function of the spatial coordinate and time.[9,10] The complex dilatational (compressional) and shear (transverse) wave speeds are given by

$$c_d = \left[\frac{\lambda+2\mu}{\rho}\right]^{1/2} \tag{6}$$

and

$$c_s = \left[\frac{\mu}{\rho}\right]^{1/2}. \tag{7}$$

If equation (3) is written in cylindrical coordinates, then

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_d^2}\frac{\partial^2 \phi}{\partial t^2}, \tag{8}$$

where $r$, $\theta$, and $z$ are the spatial coordinates in the radial, circumferential, and axial directions, respectively. If equation (2) is used, then the radial displacement is written as

$$u_r = (\nabla \phi + \nabla \times \psi)_r = \frac{\partial \phi}{\partial r} + \left( \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z} \right), \tag{9}$$

where the radial displacement $u_r$ is shown in figure 2. Note that other components of the displacement vector and related stress tensors are also shown.

Similarly, the circumferential and axial displacements are written as

$$u_\theta = (\nabla \phi + \nabla \times \psi)_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \left( \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r} \right) \tag{10}$$

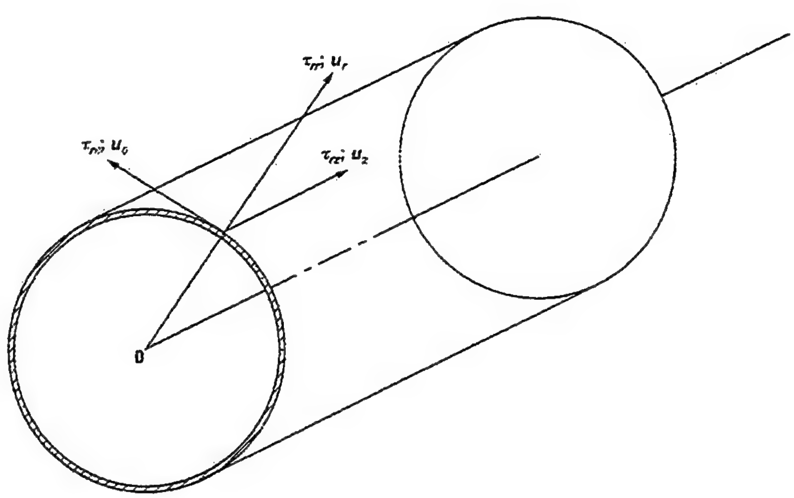


***Figure 2. Displacement Vector Components and Stress Tensors in Cylindrical Coordinates***

and

$$u_z = (\nabla\phi + \nabla\times\psi)_z$$

$$= \frac{\partial\phi}{\partial z} + \left(\frac{\psi_\theta}{r} + \frac{\partial\psi_\theta}{\partial r} - \frac{1}{r}\frac{\partial\psi_r}{\partial\theta}\right). \tag{11}$$

Equation (4) can be decomposed in the radial, circumferential, and axial directions. In the radial direction, one obtains

$$\nabla^2\psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial\psi_\theta}{\partial\theta} = \frac{1}{c_s^2}\frac{\partial^2\psi_r}{\partial t^2}$$

$$\left\{\left(\frac{\partial^2\psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_r}{\partial\theta^2} + \frac{\partial^2\psi_r}{\partial z^2}\right) - \frac{\psi_r}{r^2}\right\} - \frac{2}{r^2}\frac{\partial\psi_\theta}{\partial\theta} = \frac{1}{c_S^2}\frac{\partial^2\psi_r}{\partial t^2}. \tag{12}$$

In the circumferential direction, one obtains

$$\nabla^2\psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial\psi_r}{\partial\theta} = \frac{1}{c_s^2}\frac{\partial^2\psi_\theta}{\partial t^2}$$

$$\left\{\left(\frac{\partial^2\psi_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_\theta}{\partial\theta^2} + \frac{\partial^2\psi_\theta}{\partial z^2}\right) - \frac{\psi_\theta}{r^2}\right\} + \frac{2}{r^2}\frac{\partial\psi_r}{\partial\theta} = \frac{1}{c_S^2}\frac{\partial^2\psi_\theta}{\partial t^2}. \tag{13}$$

In the axial direction, one obtains

$$\nabla^2\psi_z = \frac{1}{c_s^2}\frac{\partial^2\psi_z}{\partial t^2}$$

$$\frac{\partial^2\psi_z}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_z}{\partial\theta^2} + \frac{\partial^2\psi_z}{\partial z^2} = \frac{1}{c_s^2}\frac{\partial^2\psi_z}{\partial t^2}. \tag{14}$$

The solution of equation (8) is written as

$$\phi(r,\theta,z,\varphi,t) = \Phi(r)\Theta(\theta)e^{-i(k_z z-\omega t)}, \tag{15}$$

where $k_z = k_0 \sin\varphi$ is the wave number in the z-direction, $k_0 = \omega / c_0$ is the acoustic wave number, $\omega$ is the frequency in radians/s, and $c_0$ is the sound speed in water. Substituting equation (15) into equation (8) gives

$$\Theta\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\Theta + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}\Phi - k_z^2\Phi\Theta = -\frac{\omega^2}{c_d^2}\Phi\Theta. \tag{16}$$

Dividing equation (16) by $\Theta$, one obtains

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} + \alpha^2\Phi = -\frac{1}{r^2}\frac{\left(\frac{d^2\Theta}{d\theta^2}\right)}{\Theta}\Phi, \tag{17}$$

where $\alpha = \left(k_d^2 - k_z^2\right)^{1/2}$, and $k_d = \omega / c_d$. Dividing equation (17) by $\Phi / r^2$ gives

$$r^2\frac{\frac{d^2\Phi}{dr^2}}{\Phi} + r\frac{\frac{d\Phi}{dr}}{\Phi} + \alpha^2 r^2 = -\frac{\frac{d^2\Theta}{d\theta^2}}{\Theta}. \tag{18}$$

Let

$$-\frac{\frac{d^2\Theta}{d\theta^2}}{\Theta} = k_\theta^2. \tag{19}$$

Then, multiplying equation (18) by $\Phi / r^2$, one obtains

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}+\left(\alpha^2-\frac{k_\theta^2}{r^2}\right)\Phi=0. \tag{20}$$

Equation (19) can be rewritten as

$$\frac{d^2\Theta}{d\theta^2}+k_\theta^2\Theta=0. \tag{21}$$

The solution of equation (21) is written as

$$\Theta(\theta)=A\cos(k_\theta\theta)+B\sin(k_\theta\theta). \tag{22}$$

Single-valueness requirements on $\Theta(\theta)$ give

$$k_\theta=n \quad \text{(an integer)}. \tag{23}$$

Then, equation (20) becomes

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}+\left(\alpha^2-\frac{n^2}{r^2}\right)\Phi=0. \tag{24}$$

Note that equation (24) is the Bessel equation of order $n$.

Equation (15) is now written as

$$\phi(r,\theta,z,\varphi,t)=\Phi(r)\cos n\theta\, e^{-i(k_z z-\omega t)}. \tag{25}$$

Then, the solution of equation (8) is written as

$$\phi(r,\theta,z,\varphi,t) = \left\{A_1 J_n(\alpha r) + B_1 Y_n(\alpha r)\right\} \cos n\theta \, e^{-i(k_z z - \omega t)}, \tag{26}$$

and $J_n(\alpha r)$ and $Y_n(\alpha r)$ are the first and second kinds of the Bessel function of order $n$, with the argument $\alpha r$, respectively.

Similarly, the solutions of equations (12) and (13) are assumed:

$$\psi_r(r,\theta,z,\varphi,t) = \Psi_r(r) \sin n\theta \, e^{-i(k_z z - \omega t)} \tag{27}$$

and

$$\psi_\theta(r,\theta,z,\varphi,t) = \Psi_\theta(r) \cos n\theta \, e^{-i(k_z z - \omega t)}. \tag{28}$$

Substituting equations (27) and (28) into equation (12) gives

$$\frac{d^2\Psi_r}{dr^2} + \frac{1}{r}\frac{d\Psi_r}{dr} + \frac{1}{r^2}\left(-n^2\Psi_r - \Psi_r + 2n\Psi_\theta\right) + \beta^2\Psi_r = 0, \tag{29}$$

where $\beta = \left(k_s^2 - k_z^2\right)^{1/2}$ and $k_s = \omega / c_s$.

Similarly, substituting equations (27) and (28) into equation (13) gives

$$\frac{d^2\Psi_\theta}{dr^2} + \frac{1}{r}\frac{d\Psi_\theta}{dr} + \frac{1}{r^2}\left(-n^2\Psi_\theta - \Psi_\theta + 2n\Psi_r\right) + \beta^2\Psi_\theta = 0. \tag{30}$$

Subtracting equation (30) from equation (29) gives

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left[\beta^2 - \frac{(n+1)^2}{r^2}\right]\right\}\left(\Psi_r - \Psi_\theta\right) = 0. \tag{31}$$

Adding equation (30) to equation (29) gives

$$\left\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\left[\beta^2-\frac{(n-1)^2}{r^2}\right]\right\}(\Psi_r+\Psi_\theta)=0. \tag{32}$$

The solution of equation (31) is written as

$$\Psi_r-\Psi_\theta=2A_2J_{n+1}(\beta r)+2B_2Y_{n+1}(\beta r). \tag{33}$$

Similarly, the solution of equation (32) is written as

$$\Psi_r+\Psi_\theta=2A_2^{'}J_{n-1}(\beta r)+2B_2^{'}Y_{n-1}(\beta r). \tag{34}$$

The property of the gauge invariance[9,10,11] can now be utilized to eliminate two of the integration constants. It may be shown that any one of three potentials, $\Phi$, $(\Psi_r - \Psi_\theta)$, and $(\Psi_r + \Psi_\theta)$, can be set equal to zero, without loss of the generality of solution. If $(\Psi_r + \Psi_\theta) = 0$, one obtains

$$\Psi_r=-\Psi_\theta, \tag{35}$$

which also means that $A_2^{'}=B_2^{'}=0$.

Substituting equation (35) into equation (33) gives

$$\Psi_r(r)=A_2J_{n+1}(\beta r)+B_2Y_{n+1}(\beta r). \tag{36}$$

Note that $\Psi_\theta(r)=-\Psi_r(r)$; i.e.,

$$\Psi_\theta(r)=-A_2J_{n+1}(\beta r)-B_2Y_{n+1}(\beta r). \tag{37}$$

If equation (36) is substituted into equation (27), then the radial component of the vector potential is written as

$$\psi_r(r,\theta,z,\varphi,t) = \{A_2 J_{n+1}(\beta r) + B_2 Y_{n+1}(\beta r)\} \sin n\theta \, e^{-i(k_z z - \omega t)}. \tag{38}$$

Similarly, the circumferential component of the vector potential is written as

$$\psi_\theta(r,\theta,z,\varphi,t) = \{-[A_2 J_{n+1}(\beta r) + B_2 Y_{n+1}(\beta r)]\} \cos n\theta \, e^{-i(k_z z - \omega t)}. \tag{39}$$

Finally, the axial component of the vector potential is assumed to be

$$\psi_z(r,\theta,z,\varphi,t) = \Psi_z(r) \sin n\theta \, e^{-i(k_z z - \omega t)}. \tag{40}$$

Substituting equation (40) into equation (14) gives

$$\Psi_z(r) = A_3 J_n(\beta r) + B_3 Y_n(\beta r). \tag{41}$$

Combining equation (41) with equation (40) gives

$$\psi_z(r,\theta,z,\varphi,t) = \{A_3 J_n(\beta r) + B_3 Y_n(\beta r)\} \sin n\theta \, e^{-i(k_z z - \omega t)}. \tag{42}$$

If equations (26), (39), and (42) are substituted into equation (9), the radial displacement is written as

$$u_r(r,\theta,z,\varphi,t)=\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}-\frac{\partial\psi_\theta}{\partial z}$$

$$=\left\{A_1\left[\alpha J_n^{'}(\alpha r)\right]+B_1\left[\alpha Y_n^{'}(\alpha r)\right]\right.$$

$$+A_2\left[-ik_z J_{n+1}(\beta r)\right]+B_2\left[-ik_z Y_{n+1}(\beta r)\right]$$

$$\left.+A_3\left[\frac{n}{r}J_n(\beta r)\right]+B_3\left[\frac{n}{r}Y_n(\beta r)\right]\right\} \tag{43}$$

$$\times\cos n\theta\, e^{-i(k_z z-\omega t)}.$$

If equations (26), (38), and (42) are substituted into equation (10), the circumferential displacement is written as

$$u_\theta(r,\theta,z,\varphi,t)=\frac{1}{r}\frac{\partial\phi}{\partial\theta}+\frac{\partial\psi_r}{\partial z}-\frac{\partial\psi_z}{\partial r}$$

$$=\left\{A_1\left[-\frac{n}{r}J_n(\alpha r)\right]+B_1\left[-\frac{n}{r}Y_n(\alpha r)\right]\right.$$

$$+A_2\left[-ik_z J_{n+1}(\beta r)\right]+B_2\left[-ik_z Y_{n+1}(\beta r)\right]$$

$$\left.+A_3\left[-\beta J_n^{'}(\beta r)\right]+B_3\left[-\beta Y_n^{'}(\beta r)\right]\right\} \tag{44}$$

$$\times\sin n\theta\, e^{-i(k_z z-\omega t)}.$$

If equations (26), (38), and (39) are substituted into equation (11), the axial displacement is written as

$$u_z(r,\theta,z,\varphi,t) = \frac{\partial\phi}{\partial z} + \frac{\psi_\theta}{r} + \frac{\partial\psi_\theta}{\partial r} - \frac{1}{r}\frac{\partial\psi_r}{\partial\theta}$$

$$= \Bigg\{ A_1\left[-ik_z J_n(\alpha r)\right] + B_1\left[-ik_z Y_n(\alpha r)\right]$$

$$+ A_2\left[-\left(\frac{n+1}{r}J_{n+1}(\beta r) + \beta J'_{n+1}(\beta r)\right)\right]$$

$$+ B_2\left[-\left(\frac{n+1}{r}Y_{n+1}(\beta r) + \beta Y'_{n+1}(\beta r)\right)\right]\Bigg\}$$

$$\times \cos n\theta \, e^{-i(k_z z-\omega t)}. \qquad (45)$$

The radial, circumferential, and axial stress components are written as

$$\tau_{rr} = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_r}{\partial r}$$

$$= (\lambda+2\mu)\left(\frac{\partial u_r}{\partial r}\right) + \left[(\lambda+2\mu)-2\mu\right]\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right), \qquad (46)$$

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \qquad (47)$$

and

$$\tau_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right). \qquad (48)$$

Substituting equations (43), (44), and (45) into equation (46), one obtains the radial stress

$$\tau_{rr}(r,\theta,z,\varphi,t) = \rho c_d^2\left(\frac{\partial u_r}{\partial r}\right)$$

$$+\rho\left(c_d^2-2c_s^2\right)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right)$$

$$=\left[A_1\left\{\rho c_d^2\cdot\alpha^2 J_n''(\alpha r)\right.\right.$$

$$\left.+\rho\left(c_d^2-2c_s^2\right)\left[-\left\{\left(\frac{n}{r}\right)^2+k_z^2\right\}J_n(\alpha r)+\frac{\alpha}{r}J_n'(\alpha r)\right]\right\}$$

$$+B_1\left\{\rho c_d^2\cdot\alpha^2 Y_n''(\alpha r)\right.$$

$$\left.+\rho\left(c_d^2-2c_s^2\right)\left[-\left\{\left(\frac{n}{r}\right)^2+k_z^2\right\}Y_n(\alpha r)+\frac{\alpha}{r}Y_n'(\alpha r)\right]\right\}$$

$$+A_2\left\{\rho c_d^2\left(-ik_z\beta J_{n+1}'(\beta r)\right)\right.$$

$$\left.+\rho\left(c_d^2-2c_s^2\right)\left[ik_z\beta J_{n+1}'(\beta r)\right]\right\}$$

$$+B_2\left\{\rho c_d^2\left(-ik_z\beta Y_{n+1}^{'}(\beta r)\right)\right.$$

$$\left.+\rho\left(c_d^2-2c_s^2\right)\left[ik_z\beta Y_{n+1}^{'}(\beta r)\right]\right\}$$

$$+A_3\left\{\rho c_d^2\left(-\frac{n}{r^2}J_n(\beta r)+\frac{n}{r}\beta J_n^{'}(\beta r)\right)\right.$$

$$\left.+\rho\left(c_d^2-2c_s^2\right)\left[-\frac{n\beta}{r}J_n^{'}(\beta r)+\frac{n}{r^2}J_n(\beta r)\right]\right\}$$

$$+B_3\left\{\rho c_d^2\left(-\frac{n}{r^2}Y_n(\beta r)+\frac{n}{r}\beta Y_n^{'}(\beta r)\right)\right.$$

$$\left.\left.+\rho\left(c_d^2-2c_s^2\right)\left[-\frac{n\beta}{r}Y_n^{'}(\beta r)+\frac{n}{r^2}Y_n(\beta r)\right]\right\}\right]$$

$$\times\cos n\theta e^{-i(k_z z-\omega t)} \tag{49}$$

Substituting equation (43) and (44) into equation (47), one obtains the circumferential stress

$$\tau_{r\theta}(r,\theta,z,\varphi,t)=\rho c_s^2\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$$

$$=\left[A_1\left\{\rho c_s^2\left[\frac{2n}{r^2}J_n(\alpha r)-\frac{2n\alpha}{r}J_n^{'}(\alpha r)\right]\right.\right.$$

$$+B_1\left\{\rho c_s^2\left[\frac{2n}{r^2}Y_n(\alpha r)-\frac{2n\alpha}{r}Y_n^{'}(\alpha r)\right]\right.$$

$$+A_2\left\{\rho c_s^2\left[\frac{i(n+1)k_z}{r}J_{n+1}(\beta r)-ik_z\beta J_{n+1}^{'}(\beta r)\right]\right\}$$

$$+B_2\left\{\rho c_s^2\left[\frac{i(n+1)}{r}k_z Y_{n+1}(\beta r)-ik_z\beta Y_{n+1}^{'}(\beta r)\right]\right\}$$

$$+A_3\left\{\rho c_s^2\left[-\left(\frac{n}{r}\right)^2 J_n(\beta r)-\beta^2 J_n^{''}(\beta r)-\left(-\frac{\beta}{r}J_n^{'}(\beta r)\right)\right]\right\}$$

$$\left.+B_3\left\{\rho c_s^2\left[-\left(\frac{n}{r}\right)^2 Y_n(\beta r)-\beta^2 Y_n^{''}(\beta r)-\left(-\frac{\beta}{r}Y_n^{'}(\beta r)\right)\right]\right\}\right]$$

$$\times \sin n\theta e^{-i(k_z z-\omega t)}. \tag{50}$$

Substituting equations (43) and (45) into equation (48), one obtains the axial stress

$$\tau_{rz}(r,\theta,z,\varphi,t)=\rho c_s^2\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)$$

$$=\left[A_1\left\{\rho c_s^2\left[-2ik_z\alpha J_n^{'}(\alpha r)\right]\right\}\right.$$

$$+B_1\left\{\rho c_s^2\left[-2ik_z\alpha Y_n^{'}(\alpha r)\right]\right\}$$

$$+A_2\left\{\rho c_s^2\left[\left(\frac{(n+1)}{r^2}-k_z^2\right)J_{n+1}(\beta r)-\frac{(n+1)}{r}\beta J_{n+1}^{'}(\beta r)-\beta^2 J_{n+1}^{''}(\beta r)\right]\right\}$$

$$+B_2\left\{\rho c_s^2\left[\left(\frac{(n+1)}{r^2}-k_z^2\right)Y_{n+1}(\beta r)-\frac{(n+1)}{r}\beta Y_{n+1}^{'}(\beta r)-\beta^2 Y_{n+1}^{''}(\beta r)\right]\right\}$$

$$+A_3\left\{\rho c_s^2\left[-\frac{ink_z}{r}J_n(\beta r)\right]\right\}$$

$$\left.+B_3\left\{\rho c_s^2\left[-\frac{ink_z}{r}Y_n(\beta r)\right]\right\}\right]$$

$$\times \cos n\theta e^{-i(k_z z-\omega t)}. \tag{51}$$

The divergence of $\psi$ is an arbitrary function:[9,10]

$$\nabla \cdot \psi = \frac{\partial \psi_r}{\partial r} + \frac{\psi_r}{r} + \frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta} + \frac{\partial \psi_z}{\partial z}. \tag{52}$$

In the outside fluid medium (water), the acoustic pressure is given by the sum of the incident wave and the scattered wave pressures as follows:

$$p_0(r,\theta,z,\varphi,t) = p_i(r,\theta,z,\varphi,t) + p_s(r,\theta,z,\varphi,t), \tag{53}$$

where $p_0(r,\theta,z,\varphi,t)$ is the total wave pressure, $p_i(r,\theta,z,\varphi,t)$ is the incident wave pressure, and $p_s(r,\theta,z,\varphi,t)$ is the scattered wave pressure. When a plane wave is incident at an angle $\varphi$ with the normal to the cylindrical axis, the total pressure is written as

$$p_0(r,\theta,z,\varphi,t) = P_i e^{-i(k_z z-\omega t)} \sum_{n=0}^{\infty} (-i)^n \varepsilon_n \left\{ J_n\left(k_0 r\cos\varphi\right) + A_0^{(n)} H_n^{(2)}\left(k_0 r\cos\varphi\right) \right\} \cos n\theta, \tag{54}$$

where $P_i$ is the amplitude of the incident wave, $A_0^{(n)}$ is the unknown coefficient to be determined for order $n$, and $\varepsilon_n$ is the Neumann constant ($\varepsilon_n = 1$ for $n=0$ and $\varepsilon_n = 2$ for $n \geq 1$). Note that the scattered outgoing wave that satisfies the radiation condition is given by the Hankel function of the second kind:

$$H_n^{(2)}(k_0 r\cos\varphi) = J_n(k_0 r\cos\varphi) - iY_n(k_0 r\cos\varphi). \tag{55}$$

Inside the coated cylindrical shell (air), the pressure field is written as

$$p_3(r,\theta,z,\varphi,t) = P_i e^{-i(k_z z-\omega t)} \sum_{n=0}^{\infty} (-i)^n \varepsilon_n B_0^{(n)} J_n\left(k_3 r\cos\varphi_3\right) \cos n\theta, \tag{56}$$

where $p_3(r, \theta, z, \varphi, t)$ is the pressure field in the core of the cylindrical structure, $B_0^{(n)}$ is the unknown coefficient to be determined for order $n$, $k_3 = \omega / c_3$, $c_3$ is the sound speed in the core, and $\cos\varphi_3 = [1 - (k_0 \sin\varphi / k_3)^2]^{1/2}$.

To ensure consistency with equations (54) and (56), the scalar potential $\phi(r, \theta, z, \varphi, t)$ and the three components of the vector potential $\psi(r,\theta,z,\varphi,t)$ may be rewritten as

$$\phi(r,\theta,z,\varphi,t) = P_i e^{-i(k_z z-\omega t)} \sum_{n=0}^{\infty} (-i)^n \varepsilon_n \left\{A_1^{(n)} J_n(\alpha r) + B_1^{(n)} Y_n(\alpha r)\right\} \cos n\theta, \tag{57}$$

$$\psi_r(r,\theta,z,\varphi,t) = P_i e^{-i(k_z z-\omega t)} \sum_{n=0}^{\infty} (-i)^n \varepsilon_n \left\{A_2^{(n)} J_{n+1}(\beta r) + B_2^{(n)} Y_{n+1}(\beta r)\right\} \sin n\theta, \tag{58}$$

$$\psi_\theta(r,\theta,z,\varphi,t) = P_i e^{-i(k_z z-\omega t)} \sum_{n=0}^{\infty} (-i)^n \varepsilon_n \left\{- A_2^{(n)} J_{n+1}(\beta r) - B_2^{(n)} Y_{n+1}(\beta r)\right\} \cos n\theta, \tag{59}$$

and

$$\psi_z(r,\theta,z,\varphi,t) = P_i e^{-i(k_z z-\omega t)} \sum_{n=0}^{\infty} (-i)^n \varepsilon_n \left\{A_3^{(n)} J_n(\beta r) + B_3^{(n)} Y_n(\beta r)\right\} \sin n\theta. \tag{60}$$

If equations (57) through (60) are used, the radial displacement $u_r$, the circumferential displacement $u_\theta$, the axial displacement $u_z$, the radial stress $\tau_{rr}$, the circumferential stress $\tau_{r\theta}$, and the axial stress $\tau_{rz}$, are preceded by the expression, $P_i \sum_{n=0}^{\infty} (-i)^n \varepsilon_n$. These six quantities include the six unknown coefficients: $A_1^{(n)}, B_1^{(n)}, A_2^{(n)}, B_2^{(n)}, A_3^{(n)}$, and $B_3^{(n)}$. These coefficients should be determined for each order $n$ by using the pertinent boundary conditions.

## FORMULATION OF THE PROBLEM

In this section, the unknown coefficients are determined by using the pertinent boundary conditions. The boundary conditions to be satisfied at the interface between the outer fluid and the outer surface of the coating (denoted by layer 1) of the cylindrical shell structure; i.e., $r = R_0$, are written as

$$\left[\tau_{rr,1}^{(n)}\right]_{r=R_0} = \left[-p_0^{(n)}\right]_{r=R_0}, \tag{61}$$

$$\left[\frac{\partial^2 u_{r,1}^{(n)}}{\partial t^2}\right]_{r=R_0} = -\frac{1}{\rho_0}\left[\frac{\partial p_0^{(n)}}{\partial r}\right]_{r=R_0}, \tag{62}$$

$$\left[\tau_{r\theta,1}^{(n)}\right]_{r=R_0} = 0, \tag{63}$$

and

$$\left[\tau_{rz,1}^{(n)}\right]_{r=R_0} = 0, \tag{64}$$

where $\left[\tau_{rr,1}^{(n)}\right]_{r=R_0}$ is the normal stress of order $n$ at the surface $r = R_0$, and the subscript 1 refers to the layer 1 (coating). Similar notations are used for the total pressure, the radial displacement, and the shear stresses. Note that $\rho_0$ is the density of the outer fluid. The boundary conditions to be satisfied at the interface between the cylindrical shell and the coating, i.e., $r = R_1$, are written as

$$\left[\tau_{rr,1}^{(n)}\right]_{r=R_1} = \left[\tau_{rr,2}^{(n)}\right]_{r=R_1}, \tag{65}$$

$$\left[\tau_{r\theta,1}^{(n)}\right]_{r=R_1}=\left[\tau_{r\theta,2}^{(n)}\right]_{r=R_1}, \tag{66}$$

$$\left[\tau_{rz,1}^{(n)}\right]_{r=R_1}=\left[\tau_{rz,2}^{(n)}\right]_{r=R_1}, \tag{67}$$

$$\left[u_{r,1}^{(n)}\right]_{r=R_1}=\left[u_{r,2}^{(n)}\right]_{r=R_1}, \tag{68}$$

$$\left[u_{\theta,1}^{(n)}\right]_{r=R_1}=\left[u_{\theta,2}^{(n)}\right]_{r=R_1}, \tag{69}$$

and

$$\left[u_{z,1}^{(n)}\right]_{r=R_1}=\left[u_{z,2}^{(n)}\right]_{r=R_1}. \tag{70}$$

The boundary conditions to be satisfied at the interface between the cylindrical shell and the core fluid, i.e., $r=R_2$, are written as

$$\left[\tau_{r\theta,2}^{(n)}\right]_{r=R_2}=0, \tag{71}$$

$$\left[\tau_{rz,2}^{(n)}\right]_{r=R_2}=0, \tag{72}$$

$$\left[\frac{\partial^2 u_{r,2}^{(n)}}{\partial t^2}\right]_{r=R_2}=-\frac{1}{\rho_3}\left[\frac{\partial p_3^{(n)}}{\partial r}\right]_{r=R_2}, \tag{73}$$

and

$$\left[\tau_{rr,2}^{(n)}\right]_{r=R_2}=\left[-p_3^{(n)}\right]_{r=R_2}, \tag{74}$$

where $p_3^{(n)}$ is the pressure field in the core, and $\rho_3$ is the density. Substituting equations (49) and (54) into equation (61) gives for order $n$

$$a_{11}^{(n)}A_0^{(n)} + a_{12}^{(n)}A_{1,1}^{(n)} + a_{13}^{(n)}B_{1,1}^{(n)} + a_{14}^{(n)}A_{2,1}^{(n)} + a_{15}^{(n)}B_{2,1}^{(n)} + a_{16}^{(n)}A_{3,1}^{(n)} + a_{17}^{(n)}B_{3,1}^{(n)} = b_1^{(n)}, \tag{75}$$

where the superscript $(n)$ denotes the order $n$, and the coefficients $a_{11}^{(n)}, a_{12}^{(n)} \ldots,$ and $b_1^{(n)}$ are given in appendix A. Note that $A_{1,1}^{(n)}$ is the unknown coefficient $A_1$ of order $n$ for the layer 1 (coating). Similar notations are used for other unknown coefficients. Substituting equations (43) and (54) into equation (62) gives for order $n$

$$a_{21}^{(n)}A_0^{(n)} + a_{22}^{(n)}A_{1,1}^{(n)} + a_{23}^{(n)}B_{1,1}^{(n)} + a_{24}^{(n)}A_{2,1}^{(n)} + a_{25}^{(n)}B_{2,1}^{(n)} + a_{26}^{(n)}A_{3,1}^{(n)} + a_{27}^{(n)}B_{3,1}^{(n)} = b_2^{(n)}. \tag{76}$$

Substituting equation (50) into equation (63) gives for order $n$

$$a_{32}^{(n)}A_{1,1}^{(n)} + a_{33}^{(n)}B_{1,1}^{(n)} + a_{34}^{(n)}A_{2,1}^{(n)} + a_{35}^{(n)}B_{2,1}^{(n)} + a_{36}^{(n)}A_{3,1}^{(n)} + a_{37}^{(n)}B_{3,1}^{(n)} = 0. \tag{77}$$

Substituting equation (51) into equation (64) gives for order $n$

$$a_{42}^{(n)}A_{1,1}^{(n)} + a_{43}^{(n)}B_{1,1}^{(n)} + a_{44}^{(n)}A_{2,1}^{(n)} + a_{45}^{(n)}B_{2,1}^{(n)} + a_{46}^{(n)}A_{3,1}^{(n)} + a_{47}^{(n)}B_{3,1}^{(n)} = 0. \tag{78}$$

Substituting equation (49) into equation (65) gives for order $n$

$$\begin{aligned} &a_{52}^{(n)}A_{1,1}^{(n)} + a_{53}^{(n)}B_{1,1}^{(n)} + a_{54}^{(n)}A_{2,1}^{(n)} + a_{55}^{(n)}B_{2,1}^{(n)} + a_{56}^{(n)}A_{3,1}^{(n)} + a_{57}^{(n)}B_{3,1}^{(n)} + a_{58}^{(n)}A_{1,2}^{(n)} \\ &\quad + a_{59}^{(n)}B_{1,2}^{(n)} + a_{5,10}^{(n)}A_{2,2}^{(n)} + a_{5,11}^{(n)}B_{2,2}^{(n)} + a_{5,12}^{(n)}A_{3,2}^{(n)} + a_{5,13}^{(n)}B_{3,2}^{(n)} = 0. \end{aligned} \tag{79}$$

Substituting equation (50) into equation (66) gives for order $n$

$$\begin{aligned} &a_{62}^{(n)}A_{1,1}^{(n)} + a_{63}^{(n)}B_{1,1}^{(n)} + a_{64}^{(n)}A_{2,1}^{(n)} + a_{65}^{(n)}B_{2,1}^{(n)} + a_{66}^{(n)}A_{3,1}^{(n)} + a_{67}^{(n)}B_{3,1}^{(n)} + a_{68}^{(n)}A_{1,2}^{(n)} \\ &\quad + a_{69}^{(n)}B_{1,2}^{(n)} + a_{6,10}^{(n)}A_{2,2}^{(n)} + a_{6,11}^{(n)}B_{2,2}^{(n)} + a_{6,12}^{(n)}A_{3,2}^{(n)} + a_{6,13}^{(n)}B_{3,2}^{(n)} = 0. \end{aligned} \tag{80}$$

Substituting equation (51) into equation (67) gives for order $n$

$$a_{72}^{(n)}A_{1,1}^{(n)}+a_{73}^{(n)}B_{1,1}^{(n)}+a_{74}^{(n)}A_{2,1}^{(n)}+a_{75}^{(n)}B_{2,1}^{(n)}+a_{76}^{(n)}A_{3,1}^{(n)}+a_{77}^{(n)}B_{3,1}^{(n)}+a_{78}^{(n)}A_{1,2}^{(n)}$$

$$+a_{79}^{(n)}B_{1,2}^{(n)}+a_{7,10}^{(n)}A_{2,2}^{(n)}+a_{7,11}^{(n)}B_{2,2}^{(n)}+a_{7,12}^{(n)}A_{3,2}^{(n)}+a_{7,13}^{(n)}B_{3,2}^{(n)}=0. \tag{81}$$

Substituting equation (43) into equation (68) gives for order $n$

$$a_{82}^{(n)}A_{1,1}^{(n)}+a_{83}^{(n)}B_{1,1}^{(n)}+a_{84}^{(n)}A_{2,1}^{(n)}+a_{85}^{(n)}B_{2,1}^{(n)}+a_{86}^{(n)}A_{3,1}^{(n)}+a_{87}^{(n)}B_{3,1}^{(n)}+a_{88}^{(n)}A_{1,2}^{(n)}$$

$$+a_{89}^{(n)}B_{1,2}^{(n)}+a_{8,10}^{(n)}A_{2,2}^{(n)}+a_{8,11}^{(n)}B_{2,2}^{(n)}+a_{8,12}^{(n)}A_{3,2}^{(n)}+a_{8,13}^{(n)}B_{3,2}^{(n)}=0. \tag{82}$$

Substituting equation (44) into equation (69) gives for order $n$

$$a_{92}^{(n)}A_{1,1}^{(n)}+a_{93}^{(n)}B_{1,1}^{(n)}+a_{94}^{(n)}A_{2,1}^{(n)}+a_{95}^{(n)}B_{2,1}^{(n)}+a_{96}^{(n)}A_{3,1}^{(n)}+a_{97}^{(n)}B_{3,1}^{(n)}+a_{98}^{(n)}A_{1,2}^{(n)}$$

$$+a_{99}^{(n)}B_{1,2}^{(n)}+a_{9,10}^{(n)}A_{2,2}^{(n)}+a_{9,11}^{(n)}B_{2,2}^{(n)}+a_{9,12}^{(n)}A_{3,2}^{(n)}+a_{9,13}^{(n)}B_{3,2}^{(n)}=0. \tag{83}$$

Substituting equation (45) into equation (70) gives for order $n$

$$a_{10,2}^{(n)}A_{1,1}^{(n)}+a_{10,3}^{(n)}B_{1,1}^{(n)}+a_{10,4}^{(n)}A_{2,1}^{(n)}+a_{10,5}^{(n)}B_{2,1}^{(n)}$$

$$+a_{10,8}^{(n)}A_{1,2}^{(n)}+a_{10,9}^{(n)}B_{1,2}^{(n)}+a_{10,10}^{(n)}A_{2,2}^{(n)}+a_{10,11}^{(n)}B_{2,2}^{(n)}=0. \tag{84}$$

Substituting equation (50) into equation (71) gives for order $n$

$$a_{11,8}^{(n)}A_{1,2}^{(n)}+a_{11,9}^{(n)}B_{1,2}^{(n)}+a_{11,10}^{(n)}A_{2,2}^{(n)}$$

$$+a_{11,11}^{(n)}B_{2,2}^{(n)}+a_{11,12}^{(n)}A_{3,2}^{(n)}+a_{11,13}^{(n)}B_{3,2}^{(n)}=0. \tag{85}$$

Substituting equation (51) into equation (72) gives for order $n$

$$a_{12,8}^{(n)}A_{1,2}^{(n)}+a_{12,9}^{(n)}B_{1,2}^{(n)}+a_{12,10}^{(n)}A_{2,2}^{(n)}$$

$$+a_{12,11}^{(n)}B_{2,2}^{(n)}+a_{12,12}^{(n)}A_{3,2}^{(n)}+a_{12,13}^{(n)}B_{3,2}^{(n)}=0. \tag{86}$$

Substituting equations (43) and (56) into equation (73) gives for order $n$

$$a_{13,8}^{(n)}A_{1,2}^{(n)}+a_{13,9}^{(n)}B_{1,2}^{(n)}+a_{13,10}^{(n)}A_{2,2}^{(n)}$$

$$+a_{13,11}^{(n)}B_{2,2}^{(n)}+a_{13,12}^{(n)}A_{3,2}^{(n)}+a_{13,13}^{(n)}B_{3,2}^{(n)}+a_{13,14}^{(n)}B_0^{(n)}=0. \tag{87}$$

Substituting equations (49) and (56) into equation (74) gives for order $n$

$$a_{14,8}^{(n)}A_{1,2}^{(n)}+a_{14,9}^{(n)}B_{1,2}^{(n)}+a_{14,10}^{(n)}A_{2,2}^{(n)}$$

$$+a_{14,11}^{(n)}B_{2,2}^{(n)}+a_{14,12}^{(n)}A_{3,2}^{(n)}+a_{14,13}^{(n)}B_{3,2}^{(n)}+a_{14,14}^{(n)}B_{0}^{(n)}=0. \tag{88}$$

Using equations (75) through (88), one may obtain a system of linear algebraic equations to be solved for the unknown coefficients for order $n$: $A_0^{(n)}$, $A_{1,1}^{(n)}$, $B_{1,1}^{(n)}$, $A_{2,1}^{(n)}$, $B_{2,1}^{(n)}$, $A_{3,1}^{(n)}$, $B_{3,1}^{(n)}$, $A_{1,2}^{(n)}$, $B_{1,2}^{(n)}$, $A_{2,2}^{(n)}$, $B_{2,2}^{(n)}$, $A_{3,2}^{(n)}$, $B_{3,2}^{(n)}$, and $B_0^{(n)}$ (shown in appendix B). After the coefficient $A_0^{(n)}$ has been obtained by solving the algebraic equations shown in appendix B, the pressure field scattered from the coated cylindrical shell at distance $r$ is written as

$$p_s(r,\theta,z,\varphi,t)=P_i e^{-i(k_z z-\omega t)}\sum_{n=0}^{\infty}(-i)^n\varepsilon_n A_0^{(n)}H_n^{(2)}(k_0 r\cos\varphi)\cos n\theta. \tag{89}$$

Further, the farfield pressure can be obtained by using the asymptotic form of the Hankel function. Its asymptotic form ($r\to\infty$) is given by

$$H_n^{(2)}(k_0 r\cos\varphi)\sim\left(\frac{2}{\pi k_0 r\cos\varphi}\right)^{1/2}\exp\left[-i\left(k_0 r\cos\varphi-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right]. \tag{90}$$

If equation (90) is combined with equation (89), the farfield pressure is written as

$$p_s(\theta,\varphi)=P_i\exp\left[-i\left(k_0 r\cos\varphi-k_z z-\frac{\pi}{4}\right)\right]\cdot\left(\frac{2}{\pi k_0 r\cos\varphi}\right)^{1/2}\sum_{n=0}^{\infty}\varepsilon_n A_0^{(n)}\cos n\theta. \tag{91}$$

## NUMERICAL CALCULATIONS AND DISCUSSION

The numerical calculations for the scattered field from the coated cylindrical shell have been made using equation (91). The normalized farfield pressure level as a function of $\theta$ for a given $\varphi$ is expressed as

$$\Psi(\theta,\varphi) = 20\log_{10}\left[\frac{\Phi(\theta,\varphi)}{\Phi(\theta,\varphi)_{\max}}\right] \text{ dB}, \tag{92}$$

where

$$\Phi(\theta,\varphi) = P_i\left(\frac{2}{\pi k_0 r\cos\varphi}\right)^{1/2}\left|\sum_{n=0}^{+\infty}\varepsilon_n A_0^{(n)}\cos n\theta\right|. \tag{93}$$

$\Psi(\theta,\varphi)$ is the directivity pattern for the coated cylindrical shell, $A_0^{(n)}$ is obtained from the solution of the equation shown in appendix B, and $\Phi(\theta,\varphi)_{\max}$ is the maximum value of $\Phi(\theta,\varphi)$ calculated for $\theta = 0 - 360°$ using equation (93). The directivity pattern for a rigid cylinder can be obtained from equation (92) provided that the coefficient $A_0^{(n)}$ is given by

$$A_0^{(n)} = -\frac{J_n'(k_0R)}{H_n^{(2)'}(k_0R)}, \tag{94}$$

where $R$ is the radius of a rigid cylinder.[3,4]

In calculating the directivity patterns using equation (92), it is necessary to use the properties of the coating and cylindrical shell, as well as their dimensions. The major parameters required in the present study are the material density, the dilatational (compressional) and shear (transverse) wave speeds in the material, and the densities and acoustic wave speeds of the fluid

media in contact with the outer and inner surfaces of the cylindrical shell structure. Note that dilatational and shear wave speeds are given by equations (6) and (7). The Lamé constants $\lambda$ and $\mu$ shown in both equations can be expressed in terms of elastic constants as follows:

$$\lambda = E\sigma/[(1+\sigma)(1-2\sigma)] \text{ and } \mu = E/[2(1+\sigma)], \tag{95}$$

where $E$ is Young's modulus and $\sigma$ is Poisson's ratio. The elastomer coating (rubber-like material) has the Young's modulus ranging on the order of $10^7$ to $10^9$ dynes/cm$^2$ as a function of frequency. The Poisson's ratio $\sigma$ for the rubber material normally approaches 0.5. Rubber is a lossy material that has an imaginary part of the Young's modulus. The complex (dynamic) Young's modulus is given by $E = E_r + iE_i$, where $E_r$ and $E_i$ are the real and imaginary parts of the Young's modulus, respectively. Actual values of $E_r$ and $E_i$ for a given material can be obtained from measurements. The ratio of $E_i$ to $E_r$ for rubber ranges from approximately 0.1 to 1.0, depending on the frequency. As can be seen in equation (95), $\lambda$ cannot be defined for rubber-like materials because $\lambda$ is infinite when $\sigma = 0.5$. Therefore, the dilatational wave speed ($c_d$) cannot be obtained from equation (6), but the shear wave speed ($c_s$) can be obtained from equation (7). An alternative expression for the dilatational wave speed in rubber is written as

$$c_d = [(B + 4\mu/3)/\rho]^{1/2}, \tag{96}$$

where $B$ is the complex bulk modulus of material. Note that $|B| >> |\mu|$. The loss factor associated with the dilatational wave is normally very small and is weakly dependent on frequency. The complex dilatational and shear wave speeds for the layer 1 (coating) are now written as follows:

$$c_{d1} = c_{d01}(1 + i\zeta_{d1})^{1/2} \tag{97}$$

and

$$c_{s1} = c_{s01}(1 + i\zeta_{s1})^{1/2}, \tag{98}$$

where $c_{d01}$ and $c_{s01}$ are the dilatational and shear wave speeds in the coating, and $\zeta_{d1}$ and $\zeta_{s1}$ are the loss factors associated with dilatational and shear waves, respectively. The cylindrical shell used in the present study is steel. The Lamé constants $\lambda$ and $\mu$ for the cylindrical shell can be expressed in terms of $E$ and $\sigma$ as shown in equation (95). Normally, the Young's modulus of steel is real because it is not considered to be lossy material. However, all real structures possess some inherent structural damping. Thus, a frequency-independent small loss factor can be assigned in the numerical calculation. Then, the complex dilatational and shear wave speeds in the cylindrical shell can be directly obtained by using equations (6) and (7) in terms of the complex Young's modulus and Poisson's ratio and are written as follows:

$$c_{d2} = c_{d02}(1 + i\zeta_{d2})^{1/2}, \tag{99}$$

and

$$c_{s2} = c_{s02}(1 + i\zeta_{s2})^{1/2}, \tag{100}$$

where $c_{d02}$ and $c_{s02}$ are the dilatational and shear wave speeds in the layer 2 (shell), and $\zeta_{d2}$ and $\zeta_{s2}$ are the loss factors associated with dilatational and shear waves, respectively. The baseline data used in the calculation of the directivity patterns are as follows:

| | |
|---|---|
| $R_2$ (inner radius of cylindrical shell) | 25.4 cm |
| $h_2$ (shell thickness) | 5.08 cm |
| $h_1$ (coating thickness) | 5.08 cm |
| $\rho_0$ (water density) | 1.0 g/cm$^3$ |
| $c_0$ (sound speed in water) | 150 000 cm/s |
| $\rho_3$ (air density) | 0.00121 g/cm$^3$ |
| $c_3$ (sound speed in air) | 34 000 cm/s |

| | |
|---|---|
| $\rho_1$ (coating density) | 0.6 g/cm$^3$ |
| $c_{s01}$ (shear wave speed in the coating) | 5000 cm/s |
| $c_{d01}$ (dilatational wave speed in the coating) | 20000 cm/s |
| $\zeta_{s1}$ (shear loss factor of the coating) | 0.3 |
| $\zeta_{d1}$ (dilatational loss factor of the coating) | 0.03 |
| $\rho_2$ (shell density) | 7.8 g/cm$^3$ |
| $E_2$ (Young's modulus of shell) | 19.5 x 10$^{11}$ dyn/cm$^2$ |
| $\sigma_2$ (Poisson's ratio of shell) | 0.3 |
| $\zeta_{s2}$ (shear loss factor of shell) | 0.01 |
| $\zeta_{d2}$ (dilatational loss factor of shell) | 0.001 |

Figure 3 shows a comparison between the directivity patterns calculated at $f$ = 5000 Hz using the two- and three-dimensional analyses. The solid line shows the directivity pattern calculated using the two-dimensional model and that calculated using the three-dimensional model with $\varphi = 0$. As shown in this figure, the two results are identical because the three-dimensional model with $\varphi = 0$ degenerates to the two-dimensional model. The dotted line in figure 3 shows the directivity pattern calculated for a rigid cylinder with an equivalent radius.

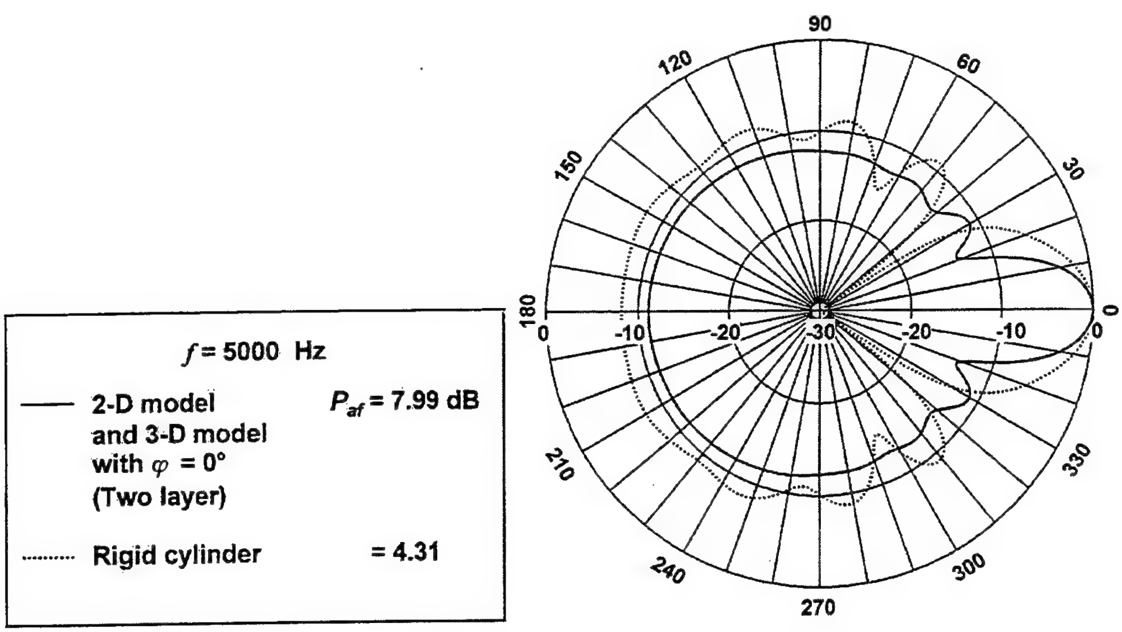


*Figure 3. Directivity Patterns at f =5000 Hz*

To compare relative scattered pressure fields for different cases (figure 3), the normalized peak response amplitude scaling factor $P_{af}$ is given in the legend for each case. This factor was calculated as follows:

$$P_{af} = 20\log_{10}\left\{\frac{\Phi(\theta,\varphi)_{\theta=0}}{P_i}\left(\frac{r}{R_0}\right)^{1/2}\right\}$$

$$= 20\log_{10}\left\{\left(\frac{2}{\pi k_0 R_0 \cos\varphi}\right)^{1/2}\left|\sum_{n=0}^{+\infty}\varepsilon_n A_0^{(n)}\right|\right\} dB. \tag{101}$$

Figures 4 through 12 present directivity patterns calculated for the coated cylindrical shell using the three-dimensional model. Figure 4 shows the directivity patterns calculated at $f = 5000$ Hz for various angles of incidence $\varphi$. In figure 4, the solid, dashed, and chain-dotted lines denote the results for $\varphi = 0°$, 30°, and 60°, respectively. As anticipated, the major lobe width of the directivity pattern becomes broader as the angle of incidence increases. In the limit, the directivity pattern becomes omnidirectional as the angle of incidence approaches 90°.

Figure 5 shows the directivity patterns calculated for the angle of incidence $\varphi = 45°$ for various frequencies. The solid, dashed, chain-dotted, and chain-dashed lines denote the results for $f = 2500$, 5000, 7500, and 10,000 Hz, respectively. It is shown in this figure that the major lobe width of the directivity pattern becomes broader as the frequency decreases. In the limit, the directivity pattern becomes omnidirectional as the frequency decreases.

Figure 6 presents the effect of the coating thickness $h_1$ on the directivity pattern. The solid, dashed, chain-dotted, and chain-dashed lines denote the results calculated at $f = 5000$ Hz and $\varphi = 45°$ for $h_1 = 2.54$, 5.08, 7.62, and 10.16 cm, respectively. It is shown in figure 6 that the difference between the results is not significant, although the results show some different values for different coating thicknesses. Figures 7 and 8 present the results similar to those shown in figure 6. Figures 7 and 8 show the directivity patterns for $f = 7500$ Hz and $f = 10{,}000$ Hz,

respectively. It is observed in both figures 7 and 8 that the effects of the coating thicknesses on the directivity patterns are not significant.

Figures 9 through 12 present the effects of the coating material parameters on the directivity patterns calculated for $f = 5000$ Hz and $\varphi = 45°$. Figure 9 shows the effect of the dilatational wave speed ($c_{d01}$) in the coating on the directivity pattern. The solid, dashed, and chain-dotted lines denote the calculated results for the dilatational wave speed $c_{d01} = 2000$, 20,000, and 200,000 cm/s, respectively. As shown in figure 9, the major lobe width becomes narrower as the dilatational wave speed becomes lower. Figure 10 shows the effect of the loss factor ($\zeta_{d1}$) associated with the dilatational wave speed on the directivity pattern. The solid, dashed, and chain-dotted lines denote the calculated results for $\zeta_{d1} = 0.03$, 0.3, and 0.9, respectively. As shown in figure 10, the difference between the results obtained for different loss factors is not significant. Figures 11 and 12 present the results calculated for various shear wave speeds ($c_{s01}$) and associated loss factors ($\zeta_{s1}$), respectively. In figure 11, the solid, dashed, and chain-dotted lines denote the calculated results for the shear wave speed $c_{s01} = 2500$, 5000, and 50,000 cm/s, respectively. In figure 12, the solid, dashed, and chain-dotted lines show the calculated results $\zeta_{s1} = 0.3$, 0.9, and 1.5, respectively. As shown in both figures 11 and 12, no substantial differences are noticed for various shear wave speeds and the associated loss factors, respectively.

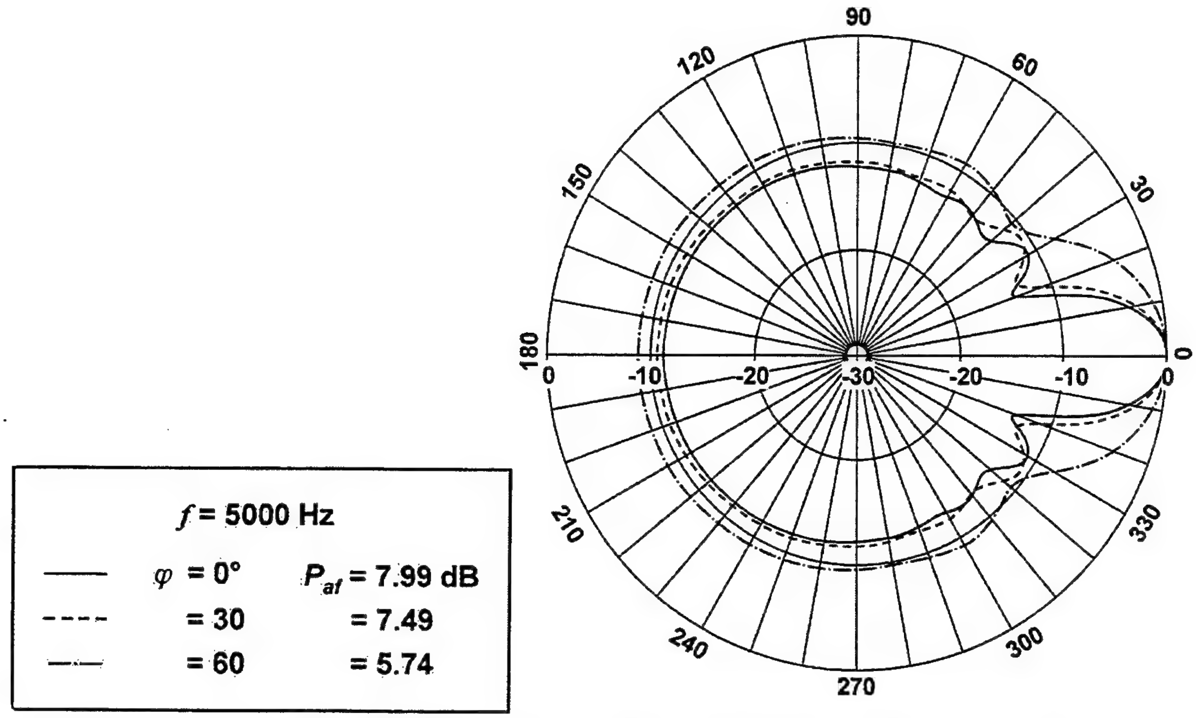


*Figure 4. Effect of Incident Angle $\varphi$ on the Directivity Pattern at $f$ = 5000 Hz*

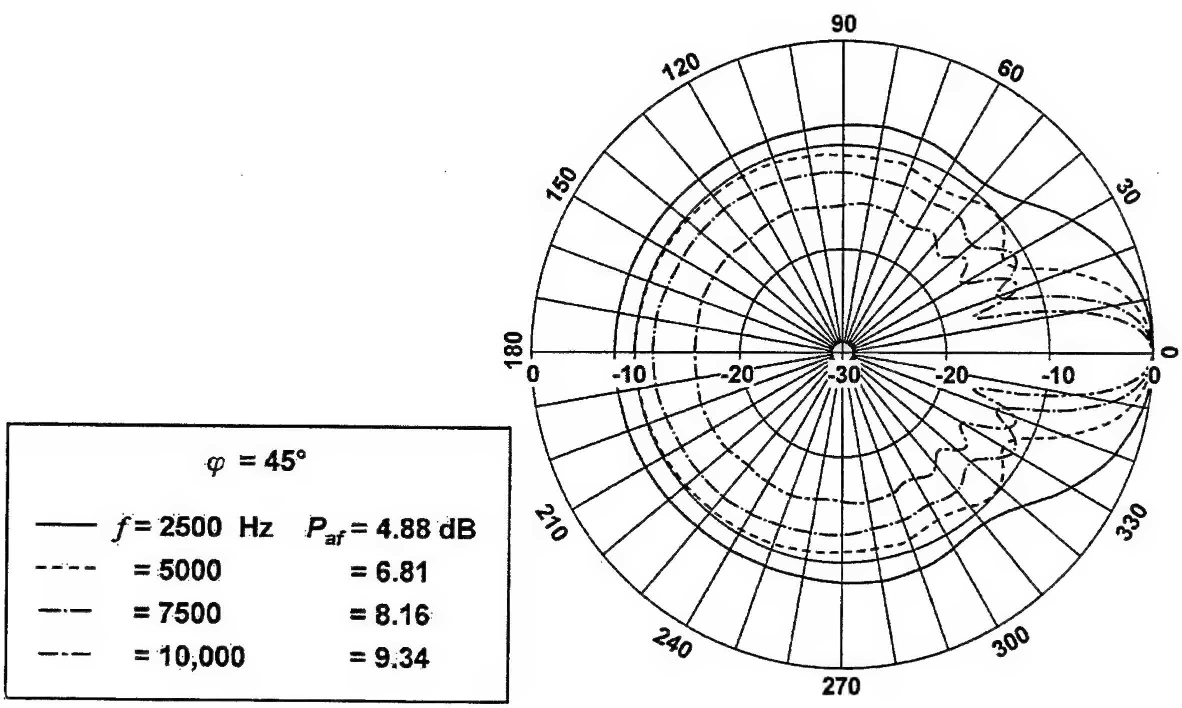


*Figure 5. Effect of Frequency $f$ on the Directivity Pattern for Incident Angle $\varphi$ = 45°*

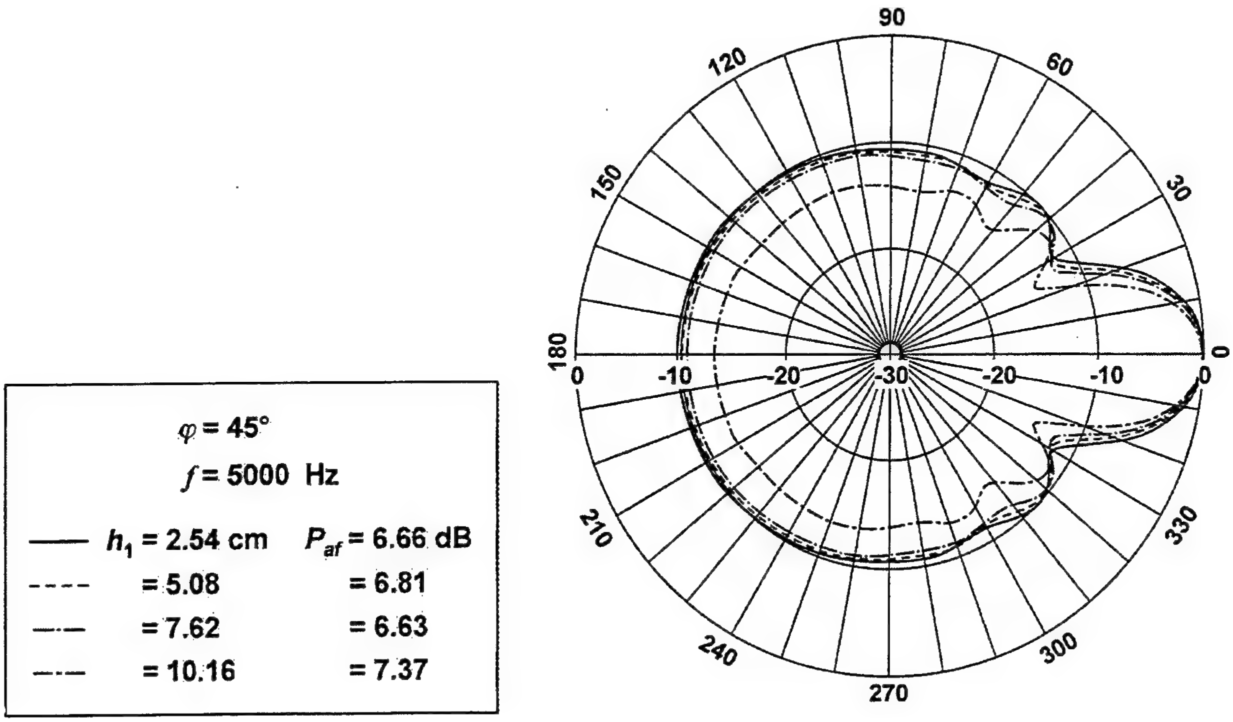


*Figure 6. Effect of Coating Thickness $h_1$ on the Directivity Pattern at $f$ = 5000 Hz for $\varphi = 45°$*

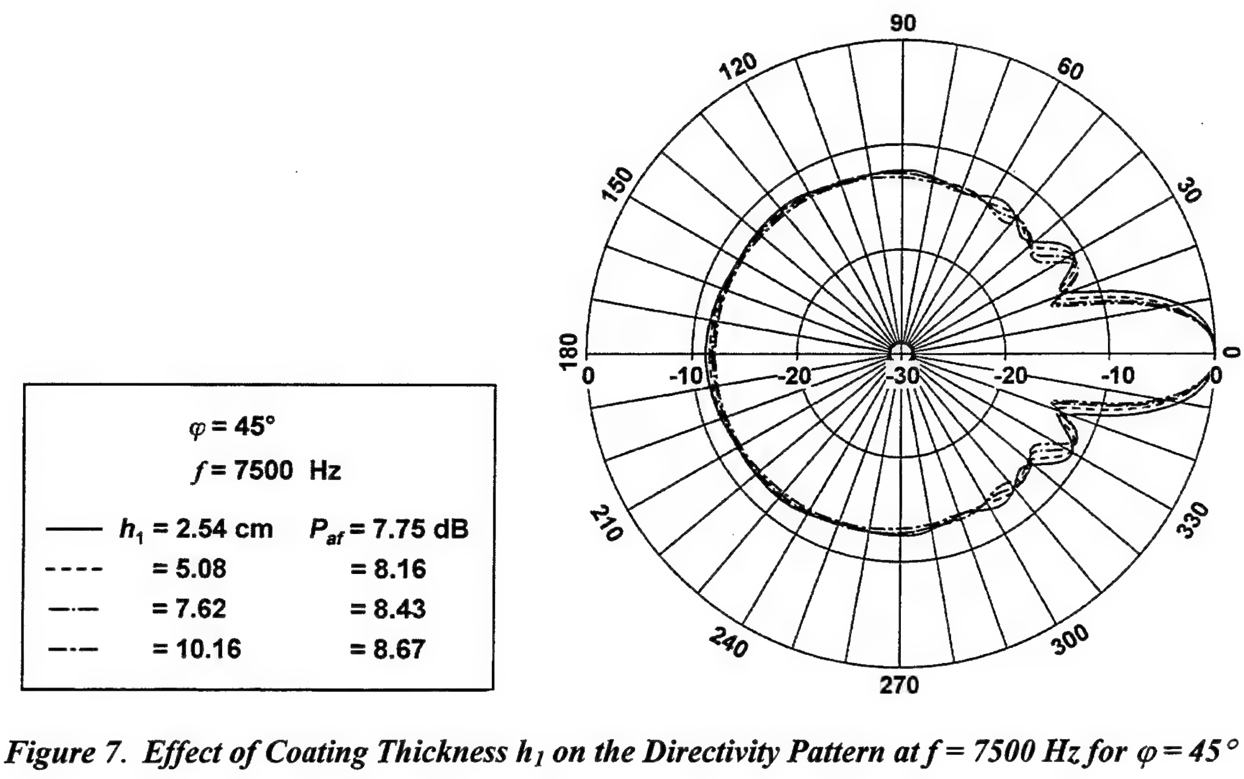


*Figure 7. Effect of Coating Thickness $h_1$ on the Directivity Pattern at $f$ = 7500 Hz for $\varphi = 45°$*

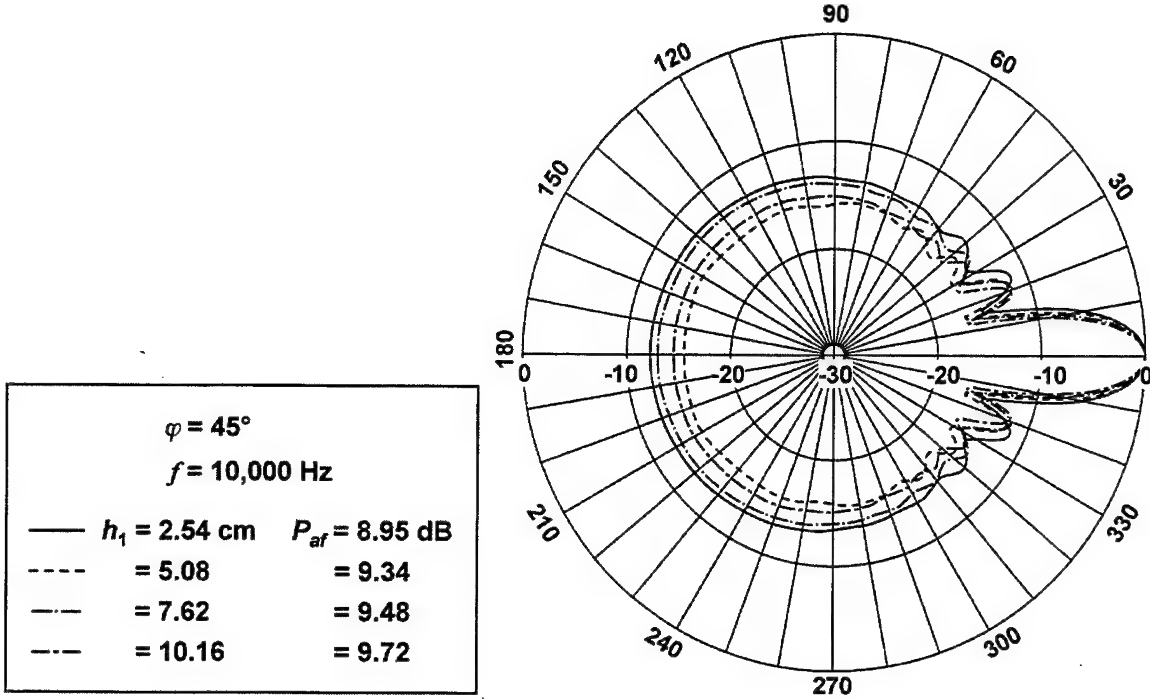


*Figure 8. Effect of Coating Thickness $h_1$ on the Directivity Pattern at $f = 10{,}000$ Hz for $\varphi = 45°$*

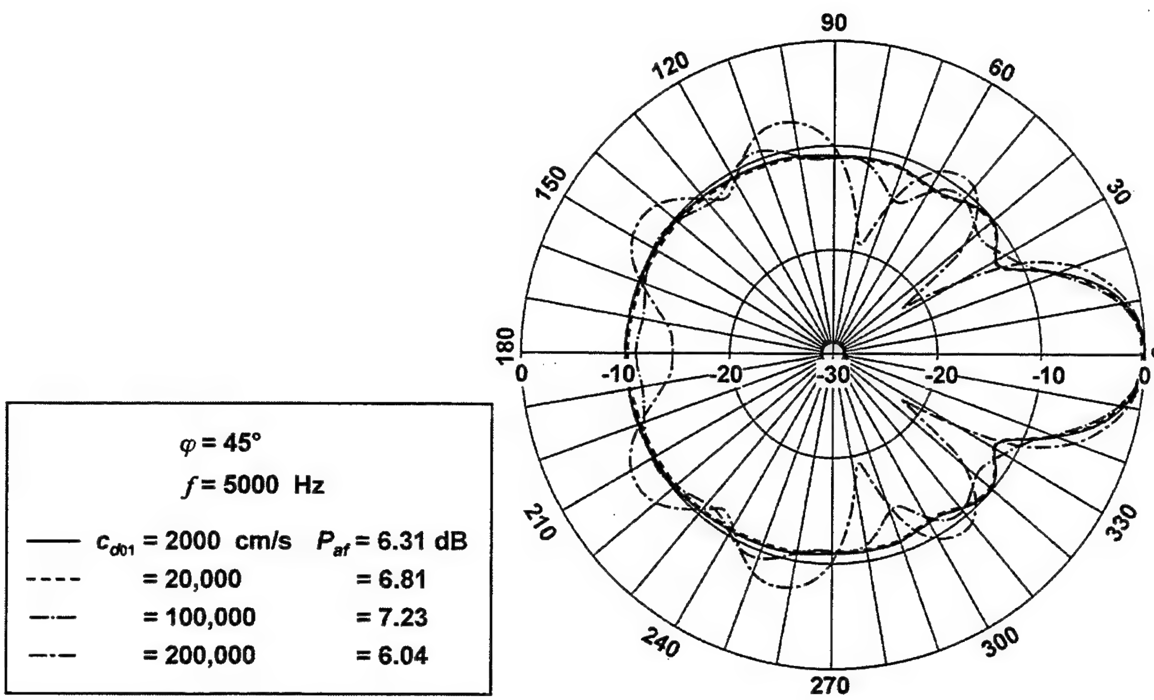


*Figure 9. Effect of Dilatational Wave Speed $c_{d01}$ on the Directivity Pattern at $f = 5000$ Hz for $\varphi = 45°$*

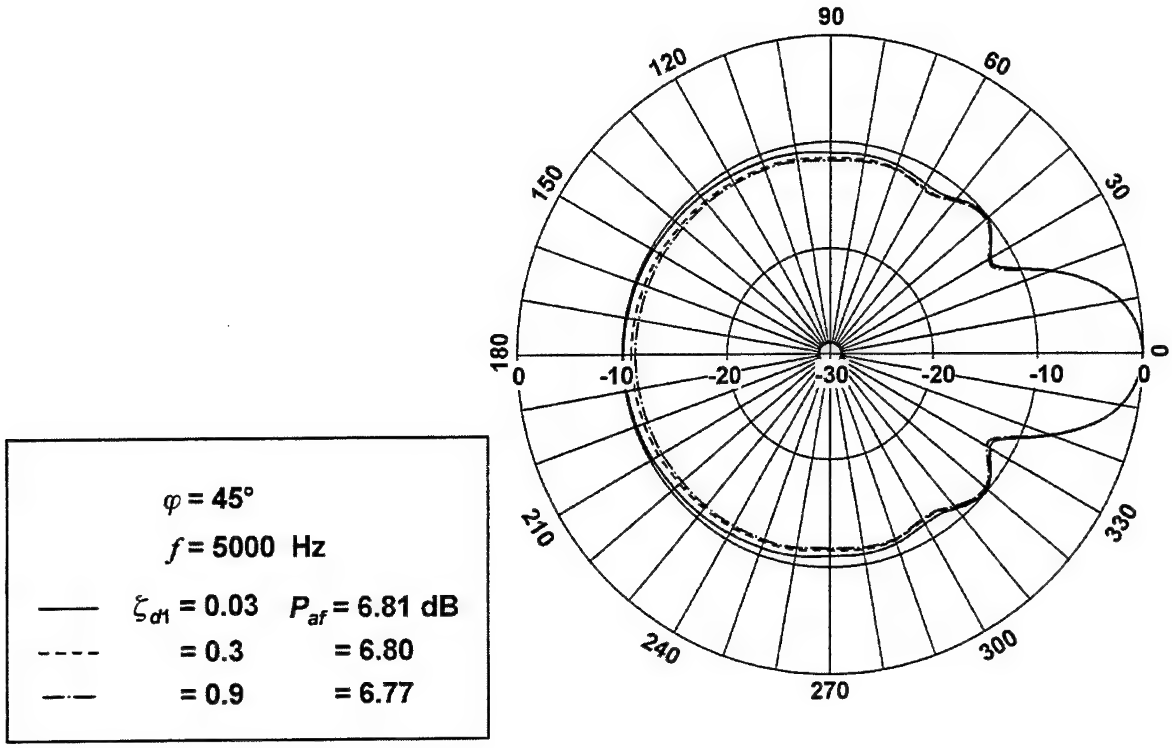


*Figure 10. Effect of Loss Factor $\zeta_{d1}$ Associated with the Dilatational Wave on the Directivity Pattern at $f$ = 5000 Hz for $\varphi$ =45 °*

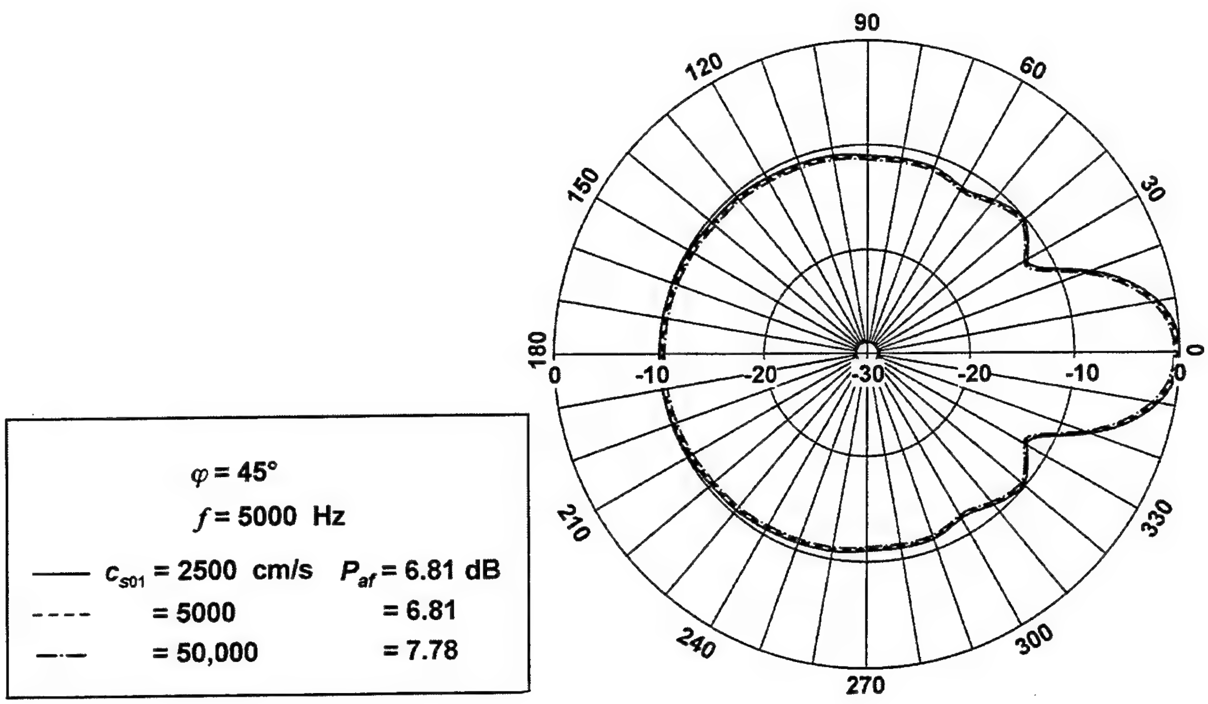


*Figure 11. Effect of Shear Wave Speed $c_{s01}$ on the Directivity Pattern at $f$ = 5000 Hz for $\varphi$ =45 °*

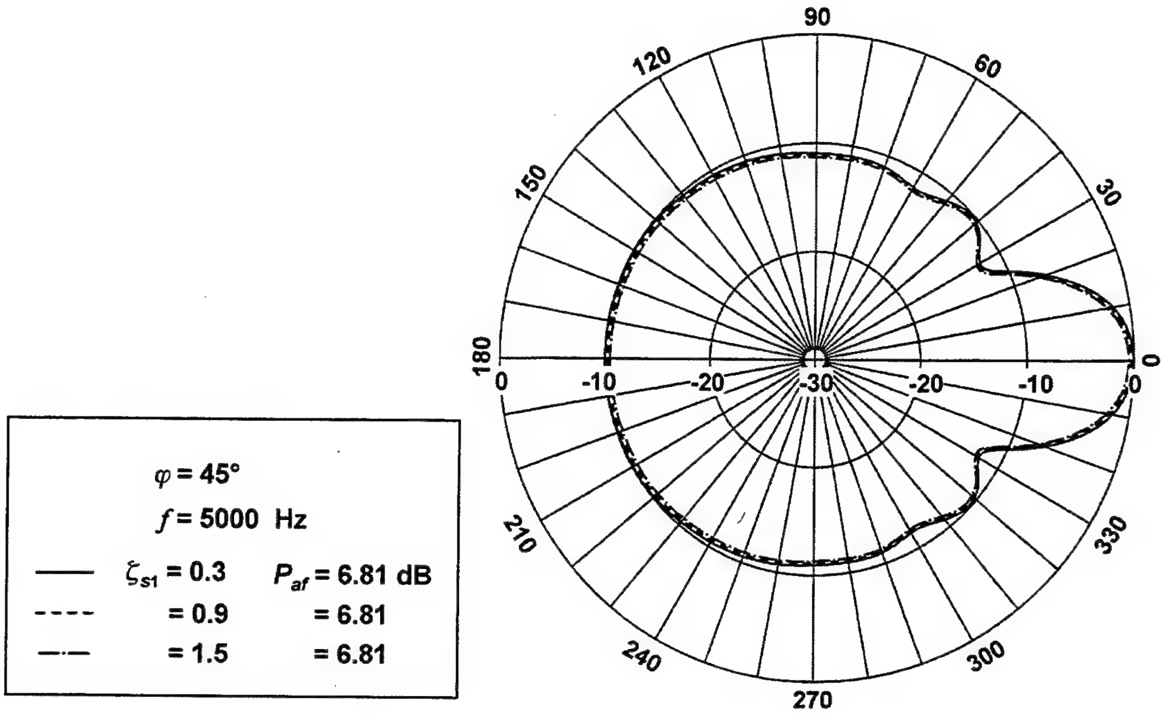


*Figure 12. Effect of Loss Factor $\zeta_{s1}$ Associated with Shear Wave on the Directivity Pattern at $f = 5000$ Hz for $\varphi = 45°$*

## CONCLUSIONS

A three-dimensional analysis has been made of the scattering from a coated cylindrical shell by a plane acoustic wave making an incident angle with the normal to the cylindrical shell axis. Based on the limited calculations of directivity patterns, the following conclusions are drawn:

1. The major lobe width of the directivity pattern becomes broader as the angle of incidence (with the normal to the axial direction) increases for a given frequency.

2. The major lobe width of the directivity pattern becomes broader as the frequency decreases for a given angle of incidence.

3. The major lobe width of the directivity pattern becomes narrower as the thickness of the coating increases; however, the difference between various thicknesses of the coating is insignificant.

4. The major lobe width becomes narrower as the dilatational wave speed in the coating becomes lower. The contribution of the loss factor associated with the dilatational wave to the directivity pattern is insignificant.

Similar conclusions are drawn for the shear waves propagating in the coating.

## REFERENCES


1. P. M. Morse, *Vibration and Sound*, McGraw-Hill Book Co., New York, 1948.

2. J. J. Bowen, J. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering from Simple Shapes*, Hemisphere Corp., New York, 1987.

3. G. C. Gaunaurd, "Sonar Cross Section of a Coated Hollow Cylinder in Water," *Journal of the Acoustical Society of America*, vol. 61, 1977, pp. 360-368.

4. L. Flax and W. G. Neubauer, "Acoustic Reflection from Layered Elastic Absorptive Cylinders," *Journal of the Acoustical Society of America*, vol. 61, 1977, pp. 301-312.

5. S. H. Ko, "Directivity Patterns for a Coated Cylindrical Shell (Two-Dimensional Analysis)," NUWC-NPT Technical Report 10,835, Naval Undersea Warfare Center Division, Newport, RI, 15 September 1995.

6. S. H. Ko and B. E. Sandman, "Acoustic Wave Scattering from a Coated Cylindrical Shell," *Journal of the Acoustical Society of America*, vol. 103, no. 2, 1998, p. 3073.

7. H. Kolsky, *Stress Wave in Solids*, Dover Publications, Inc., New York, 1963.

8. T. R. Meeker and A. H. Meizler, "Guided Wave Propagation in Elongated Cylinders and Plates," *Physical Acoustics*, W. P. Mason, ed., Academic Press, New York, 1964.

9. D. C. Gazis, "Three-Dimensional Investigation of the Propagation of Waves in Hollow Circular Cylinders, I. Analytical Foundation," *Journal of the Acoustical Society of America*, vol. 31, 1959, pp. 568-573.

10. K. F. Graff, *Wave Motion in Elastic Solids*, Dover Publications, Inc., 1975.

11. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill Book Co., New York, 1953.

# APPENDIX A
# COEFFICIENTS OF EQUATIONS (75) THROUGH (88)

$$a_{11}^{(n)} = H_n^{(2)}\left(k_0 R_0 \cos\varphi\right)$$

$$a_{12}^{(n)} = \rho_1 c_{d1}^2 \alpha_1^2 J_n''\left(\alpha_1 R_0\right) + \rho_1\left(c_{d1}^2 - 2c_{s1}^2\right)\left\{-\left[\left(\frac{n}{R_0}\right)^2 + k_z^2\right] J_n\left(\alpha_1 R_0\right) + \frac{\alpha_1}{R_0} J_n'(\alpha_1 R_0)\right\}$$

$$a_{13}^{(n)} = \rho_1 c_{d1}^2 \alpha_1^2 Y_n''\left(\alpha_1 R_0\right) + \rho_1\left(c_{d1}^2 - 2c_{s1}^2\right)\left\{-\left[\left(\frac{n}{R_0}\right)^2 + k_z^2\right] Y_n\left(\alpha_1 R_0\right) + \frac{\alpha_1}{R_0} Y_n'(\alpha_1 R_0)\right\}$$

$$a_{14}^{(n)} = \rho_1 c_{d1}^2 \left\{- i k_z \beta_1 J_{n+1}'(\beta_1 R_0)\right\} + \rho_1 \ (c_{d1}^2 - 2c_{s1}^2)\left\{+ i k_z \beta_1 J_{n+1}'(\beta_1 R_0)\right\}$$

$$a_{15}^{(n)} = \rho_1 c_{d1}^2 \left\{- i k_z \beta_1 Y_{n+1}'(\beta_1 R_0)\right\} + \rho_1 \ (c_{d1}^2 - 2c_{s1}^2)\left\{+ i k_z \beta_1 Y_{n+1}'(\beta_1 R_0)\right\}$$

$$a_{16}^{(n)} = \rho_1 c_{d1}^2 \left\{-\frac{n}{R_0^2} J_n(\beta_1 R_0) + \frac{n}{R_0}\beta_1 J_n'(\beta_1 R_0)\right\} + \rho_1 (c_{d1}^2 - 2c_{s1}^2)\left\{-\frac{n\beta_1}{R_0} J_n'(\beta_1 R_0) + \frac{n}{R_0^2} J_n(\beta_1 R_0)\right\}$$

$$a_{17}^{(n)} = \rho_1 c_{d1}^2 \left\{-\frac{n}{R_0^2} Y_n(\beta_1 R_0) + \frac{n}{R_0}\beta_1 Y_n'(\beta_1 R_0)\right\} + \rho_1 (c_{d1}^2 - 2c_{s1}^2)\left\{-\frac{n\beta_1}{R_0} Y_n'(\beta_1 R_0) + \frac{n}{R_0^2} Y_n(\beta_1 R_0)\right\}$$

$$a_{21}^{(n)} = -k_0 \cos\varphi \, H_n^{(2)'}(k_0 R_0 \cos\varphi)$$

$$a_{22}^{(n)} = \rho_0 \omega^2 \alpha_1 J_n'(\alpha_1 R_0)$$

$$a_{23}^{(n)} = \rho_0 \omega^2 \alpha_1 Y_n'(\alpha_1 R_0)$$

$$a_{24}^{(n)} = -i\rho_0\omega^2 k_z J_{n+1}(\beta_1 R_0)$$

$$a_{25}^{(n)} = -i\rho_0\omega^2 k_z Y_{n+1}(\beta_1 R_0)$$

$$a_{26}^{(n)} = \rho_0\omega^2 \frac{n}{R_0} J_n(\beta_1 R_0)$$

$$a_{27}^{(n)} = \rho_0\omega^2 \frac{n}{R_0} Y_n(\beta_1 R_0)$$

$$a_{32}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{2n}{R_0^2} J_n(\alpha_1 R_0) - \frac{2n\alpha_1}{R_0} J_n^{'}(\alpha_1 R_0) \right\}$$

$$a_{33}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{2n}{R_0^2} Y_n(\alpha_1 R_0) - \frac{2n\alpha_1}{R_0} Y_n^{'}(\alpha_1 R_0) \right\}$$

$$a_{34}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{i(n+1)}{R_0} k_z J_{n+1}(\beta_1 R_0) - ik_z\beta_1 J_{n+1}^{'}(\beta_1 R_0) \right\}$$

$$a_{35}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{i(n+1)}{R_0} k_z Y_{n+1}(\beta_1 R_0) - ik_z\beta_1 Y_{n+1}^{'}(\beta_1 R_0) \right\}$$

$$a_{36}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\left(\frac{n}{R_0}\right)^2 J_n(\beta_1 R_0) + \frac{\beta_1}{R_0} J_n^{'}(\beta_1 R_0) - \beta_1^2 J_n^{''}(\beta_1 R_0) \right\}$$

$$a_{37}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\left(\frac{n}{R_0}\right)^2 Y_n(\beta_1 R_0) + \frac{\beta_1}{R_0} Y_n^{'}(\beta_1 R_0) - \beta_1^2 Y_n^{''}(\beta_1 R_0) \right\}$$

$$a_{42}^{(n)} = \rho_1 c_{s1}^2 \left\{ -2ik_z\alpha_1 J_n^{'}(\alpha_1 R_0) \right\}$$

$$a_{43}^{(n)} = \rho_1 c_{s1}^2 \left\{ -2ik_z \alpha_1 Y_n^{'}(\alpha_1 R_0) \right\}$$

$$a_{44}^{(n)} = \rho_1 c_{s1}^2 \left\{ \left( \frac{n+1}{R_0^2} - k_z^2 \right) J_{n+1}(\beta_1 R_0) - \frac{n+1}{R_0} \beta_1 J_{n+1}^{'}(\beta_1 R_0) - \beta_1^2 J_{n+1}^{''}(\beta_1 R_0) \right\}$$

$$a_{45}^{(n)} = \rho_1 c_{s1}^2 \left\{ \left( \frac{n+1}{R_0^2} - k_z^2 \right) Y_{n+1}(\beta_1 R_0) - \frac{n+1}{R_0} \beta_1 Y_{n+1}^{'}(\beta_1 R_0) - \beta_1^2 Y_{n+1}^{''}(\beta_1 R_0) \right\}$$

$$a_{46}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\frac{ink_z}{R_0} J_n(\beta_1 R_0) \right\}$$

$$a_{47}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\frac{ink_z}{R_0} Y_n(\beta_1 R_0) \right\}$$

$$a_{52}^{(n)} = \rho_1 c_{d1}^2 \alpha_1^2 J_n^{''}(\alpha_1 R_1) + \rho_1 (c_{d1}^2 - 2c_{s1}^2) \left\{ -\left[ \left( \frac{n}{R_1} \right)^2 + k_z^2 \right] J_n(\alpha_1 R_1) + \frac{\alpha_1}{R_1} J_n^{'}(\alpha_1 R_1) \right\}$$

$$a_{53}^{(n)} = \rho_1 c_{d1}^2 \alpha_1^2 Y_n^{''}(\alpha_1 R_1) + \rho_1 (c_{d1}^2 - 2c_{s1}^2) \left\{ -\left[ \left( \frac{n}{R_1} \right)^2 + k_z^2 \right] Y_n(\alpha_1 R_1) + \frac{\alpha_1}{R_1} Y_n^{'}(\alpha_1 R_1) \right\}$$

$$a_{54}^{(n)} = \rho_1 c_{d1}^2 \left\{ -ik_z \beta_1 J_{n+1}^{'}(\beta_1 R_1) \right\} + \rho_1 (c_{d1}^2 - 2c_{s1}^2) \left\{ ik_z \beta_1 J_{n+1}^{'}(\beta_1 R_1) \right\}$$

$$a_{55}^{(n)} = \rho_1 c_{d1}^2 \left\{ -ik_z \beta_1 Y_{n+1}^{'}(\beta_1 R_1) \right\} + \rho_1 (c_{d1}^2 - 2c_{s1}^2) \left\{ ik_z \beta_1 Y_{n+1}^{'}(\beta_1 R_1) \right\}$$

$$a_{56}^{(n)} = \rho_1 c_{d1}^2 \left\{ -\frac{n}{R_1^2} J_n(\beta_1 R_1) + \frac{n}{R_1} \beta_1 J_n^{'}(\beta_1 R_1) \right\}$$

$$+\rho_1 (c_{d1}^2 - 2c_{s1}^2) \left\{ -\frac{n\beta_1}{R_1} J_n^{'}(\beta_1 R_1) + \frac{n}{R_1^2} J_n(\beta_1 R_1) \right\}$$

$$a_{57}^{(n)} = \rho_1 c_{d1}^2 \left\{ -\frac{n}{R_1^2} Y_n(\beta_1 R_1) + \frac{n}{R_1} \beta_1 Y_n^{'}(\beta_1 R_1) \right\}$$

$$+\rho_1 (c_{d1}^2 - 2c_{s1}^2) \left\{ -\frac{n\beta_1}{R_1} Y_n^{'}(\beta_1 R_1) + \frac{n}{R_1^2} Y_n(\beta_1 R_1) \right\}$$

$$a_{58}^{(n)} = -\left\{ \rho_2 c_{d2}^2 \alpha_2^2 J_n^{''}(\alpha_2 R_1) + \rho_2 (c_{d2}^2 - 2c_{s2}^2) \left[ -\left( \left( \frac{n}{R_1} \right)^2 + k_z^2 \right) J_n(\alpha_2 R_1) + \frac{\alpha_2}{R_1} J_n^{'}(\alpha_2 R_1) \right] \right\}$$

$$a_{59}^{(n)} = -\left\{ \rho_2 c_{d2}^2 \alpha_2^2 Y_n^{''}(\alpha_2 R_1) + \rho_2 (c_{d2}^2 - 2c_{s2}^2) \left[ -\left( \left( \frac{n}{R_1} \right)^2 + k_z^2 \right) Y_n(\alpha_2 R_1) + \frac{\alpha_2}{R_1} Y_n^{'}(\alpha_2 R_1) \right] \right\}$$

$$a_{5,10}^{(n)} = -\left\{ \rho_2 c_{d2}^2 \left[ -ik_z \beta_2 J_{n+1}^{'}(\beta_2 R_1) \right] + \rho_2 (c_{d2}^2 - 2c_{s2}^2) \left[ ik_z \beta_2 J_{n+1}^{'}(\beta_2 R_1) \right] \right\}$$

$$a_{5,11}^{(n)} = -\left\{ \rho_2 c_{d2}^2 \left[ -ik_z \beta_2 Y_{n+1}^{'}(\beta_2 R_1) \right] + \rho_2 (c_{d2}^2 - 2c_{s2}^2) \left[ ik_z \beta_2 Y_{n+1}^{'}(\beta_2 R_1) \right] \right\}$$

$$a_{5,12}^{(n)} = -\left\{ \rho_2 c_{d2}^2 \left[ -\frac{n}{R_1^2} J_n(\beta_2 R_1) + \frac{n}{R_1} \beta_2 J_n^{'}(\beta_2 R_1) \right] \right.$$

$$\left. +\rho_2 (c_{d2}^2 - 2c_{s2}^2) \left[ -\frac{n\beta_2}{R_1} J_n^{'}(\beta_2 R_1) + \frac{n}{R_1^2} J_n(\beta_2 R_1) \right] \right\}$$

$$a_{5,13}^{(n)} = -\left\{ \rho_2 c_{d2}^2 \left[ -\frac{n}{R_1^2} Y_n(\beta_2 R_1) + \frac{n}{R_1} \beta_2 Y_n^{'}(\beta_2 R_1) \right] \right.$$

$$\left. + \rho_2 (c_{d2}^2 - 2c_{s2}^2) \left[ -\frac{n\beta_2}{R_1} Y_n^{'}(\beta_2 R_1) + \frac{n}{R_1^2} Y_n(\beta_2 R_1) \right] \right\}$$

$$a_{62}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{2n}{R_1^2} J_n(\alpha_1 R_1) - \frac{2n\alpha_1}{R_1} J_n^{'}(\alpha_1 R_1) \right\}$$

$$a_{63}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{2n}{R_1^2} Y_n(\alpha_1 R_1) - \frac{2n\alpha_1}{R_1} Y_n^{'}(\alpha_1 R_1) \right\}$$

$$a_{64}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{i(n+1)k_z}{R_1} J_{n+1}(\beta_1 R_1) - ik_z \beta_1 J_{n+1}^{'}(\beta_1 R_1) \right\}$$

$$a_{65}^{(n)} = \rho_1 c_{s1}^2 \left\{ \frac{i(n+1)k_z}{R_1} Y_{n+1}(\beta_1 R_1) - ik_z \beta_1 Y_{n+1}^{'}(\beta_1 R_1) \right\}$$

$$a_{66}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\left( \frac{n}{R_1} \right)^2 J_n(\beta_1 R_1) + \frac{\beta_1}{R_1} J_n^{'}(\beta_1 R_1) - \beta_1^2 J_n^{''}(\beta_1 R_1) \right\}$$

$$a_{67}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\left( \frac{n}{R_1} \right)^2 Y_n(\beta_1 R_1) + \frac{\beta_1}{R_1} Y_n^{'}(\beta_1 R_1) - \beta_1^2 Y_n^{''}(\beta_1 R_1) \right\}$$

$$a_{68}^{(n)} = -\left\{ \rho_2 c_{s2}^2 \left[ \frac{2n}{R_1^2} J_n(\alpha_2 R_1) - \frac{2n\alpha_2}{R_1} J_n^{'}(\alpha_2 R_1) \right] \right\}$$

$$a_{69}^{(n)} = -\left\{\rho_2 c_{s2}^2\left[\frac{2n}{R_1^2}Y_n(\alpha_2 R_1) - \frac{2n\alpha_2}{R_1}Y_n'(\alpha_2 R_1)\right]\right\}$$

$$a_{6,10}^{(n)} = -\left\{\rho_2 c_{s2}^2\left[\frac{i(n+1)k_z}{R_1}J_{n+1}(\beta_2 R_1) - ik_z\beta_2 J_{n+1}'(\beta_2 R_1)\right]\right\}$$

$$a_{6,11}^{(n)} = -\left\{\rho_2 c_{s2}^2\left[\frac{i(n+1)k_z}{R_1}Y_{n+1}(\beta_2 R_1) - ik_z\beta_2 Y_{n+1}'(\beta_2 R_1)\right]\right\}$$

$$a_{6,12}^{(n)} = -\left\{\rho_2 c_{s2}^2\left[-\left(\frac{n}{R_1}\right)^2 J_n(\beta_2 R_1) + \frac{\beta_2}{R_1}J_n'(\beta_2 R_1) - \beta_2^2 J_n''(\beta_2 R_1)\right]\right\}$$

$$a_{6,13}^{(n)} = -\left\{\rho_2 c_{s2}^2\left[-\left(\frac{n}{R_1}\right)^2 Y_n(\beta_2 R_1) + \frac{\beta_2}{R_1}Y_n'(\beta_2 R_1) - \beta_2^2 Y_n''(\beta_2 R_1)\right]\right\}$$

$$a_{72}^{(n)} = \rho_1 c_{s1}^2\left\{-2ik_z\alpha_1 J_n'(\alpha_1 R_1)\right\}$$

$$a_{73}^{(n)} = \rho_1 c_{s1}^2\left\{-2ik_z\alpha_1 Y_n'(\alpha_1 R_1)\right\}$$

$$a_{74}^{(n)} = \rho_1 c_{s1}^2\left\{\left(\frac{n+1}{R_1^2} - k_z^2\right)J_{n+1}(\beta_1 R_1) - \frac{n+1}{R_1}\beta_1 J_{n+1}'(\beta_1 R_1) - \beta_1^2 J_{n+1}''(\beta_1 R_1)\right\}$$

$$a_{75}^{(n)} = \rho_1 c_{s1}^2\left\{\left(\frac{n+1}{R_1^2} - k_z^2\right)Y_{n+1}(\beta_1 R_1) - \frac{n+1}{R_1}\beta_1 Y_{n+1}'(\beta_1 R_1) - \beta_1^2 Y_{n+1}''(\beta_1 R_1)\right\}$$

$$a_{76}^{(n)} = \rho_1 c_{s1}^2\left\{-\frac{ink_z}{R_1}J_n(\beta_1 R_1)\right\}$$

$$a_{77}^{(n)} = \rho_1 c_{s1}^2 \left\{ -\frac{ink_z}{R_1} Y_n(\beta_1 R_1) \right\}$$

$$a_{78}^{(n)} - \left\{ \rho_2 c_{s2}^2 \left[ -2ik_z \alpha_2 J_n^{'}(\alpha_2 R_1) \right] \right\}$$

$$a_{79}^{(n)} - \left\{ \rho_2 c_{s2}^2 \left[ -2ik_z \alpha_2 Y_n^{'}(\alpha_2 R_1) \right] \right\}$$

$$a_{7,10}^{(n)} = -\left\{ \rho_2 c_{s2}^2 \left[ \left( \frac{n+1}{R_1^2} - k_z^2 \right) J_{n+1}(\beta_2 R_1) - \frac{n+1}{R_1} \beta_2 J_{n+1}^{'}(\beta_2 R_1) - \beta_2^2 J_{n+1}^{''}(\beta_2 R_1) \right] \right\}$$

$$a_{7,11}^{(n)} = -\left\{ \rho_2 c_{s2}^2 \left[ \left( \frac{n+1}{R_1^2} - k_z^2 \right) Y_{n+1}(\beta_2 R_1) - \frac{n+1}{R_1} \beta_2 Y_{n+1}^{'}(\beta_2 R_1) - \beta_2^2 Y_{n+1}^{''}(\beta_2 R_1) \right] \right\}$$

$$a_{7,12}^{(n)} = -\left\{ \rho_2 c_{s2}^2 \left[ -\frac{ink_z}{R_1} J_n(\beta_2 R_1) \right] \right\}$$

$$a_{7,13}^{(n)} = -\left\{ \rho_2 c_{s2}^2 \left[ -\frac{ink_z}{R_1} Y_n(\beta_2 R_1) \right] \right\}$$

$$a_{82}^{(n)} = \alpha_1 J_n^{'}(\alpha_1 R_1)$$

$$a_{83}^{(n)} = \alpha_1 Y_n^{'}(\alpha_1 R_1)$$

$$a_{84}^{(n)} = -ik_z J_{n+1}(\beta_1 R_1)$$

$$a_{85}^{(n)} = -ik_z Y_{n+1}(\beta_1 R_1)$$

$$a_{86}^{(n)} = \frac{n}{R_1} J_n(\beta_1 R_1)$$

$$a_{87}^{(n)} = \frac{n}{R_1} Y_n(\beta_1 R_1)$$

$$a_{88}^{(n)} = -\left\{\alpha_2 J_n^{'}(\alpha_2 R_1)\right\}$$

$$a_{89}^{(n)} = -\left\{\alpha_2 Y_n^{'}(\alpha_2 R_1)\right\}$$

$$a_{8,10}^{(n)} = -\left\{-ik_z J_{n+1}(\beta_2 R_1)\right\}$$

$$a_{8,11}^{(n)} = -\left\{-ik_z Y_{n+1}(\beta_2 R_1)\right\}$$

$$a_{8,12}^{(n)} = -\left\{\frac{n}{R_1} J_n(\beta_2 R_1)\right\}$$

$$a_{8,13}^{(n)} = -\left\{\frac{n}{R_1} Y_n(\beta_2 R_1)\right\}$$

$$a_{92}^{(n)} = -\frac{n}{R_1} J_n(\alpha_1 R_1)$$

$$a_{93}^{(n)} = -\frac{n}{R_1} Y_n(\alpha_1 R_1)$$

$$a_{94}^{(n)} = -ik_z J_{n+1}(\beta_1 R_1)$$

$$a_{95}^{(n)} = -ik_z Y_{n+1}(\beta_1 R_1)$$

$$a_{96}^{(n)} = -\beta_1 J_n'(\beta_1 R_1)$$

$$a_{97}^{(n)} = -\beta_1 Y_n'(\beta_1 R_1)$$

$$a_{98}^{(n)} = -\left\{-\frac{n}{R_1} J_n(\alpha_2 R_1)\right\}$$

$$a_{99}^{(n)} = -\left\{-\frac{n}{R_1} Y_n(\alpha_2 R_1)\right\}$$

$$a_{9,10}^{(n)} = -\left\{-ik_z J_{n+1}(\beta_2 R_1)\right\}$$

$$a_{9,11}^{(n)} = -\left\{-ik_z Y_{n+1}(\beta_2 R_1)\right\}$$

$$a_{9,12}^{(n)} = -\left\{-\beta_2 J_n'(\beta_2 R_1)\right\}$$

$$a_{9,13}^{(n)} = -\left\{-\beta_2 Y_n'(\beta_2 R_1)\right\}$$

$$a_{10,2}^{(n)} = -ik_z J_n(\alpha_1 R_1)$$

$$a_{10,3}^{(n)} = -ik_z Y_n(\alpha_1 R_1)$$

$$a_{10,4}^{(n)} = -\left\{\frac{n+1}{R_1} J_{n+1}(\beta_1 R_1) + \beta_1 J_{n+1}'(\beta_1 R_1)\right\}$$

$$a_{10,5}^{(n)} = -\left\{ \frac{n+1}{R_1} Y_{n+1}(\beta_1 R_1) + \beta_1 Y_{n+1}^{'}(\beta_1 R_1) \right\}$$

$$a_{10,8}^{(n)} = -\{ -ik_z J_n(\alpha_2 R_1) \}$$

$$a_{10,9}^{(n)} = -\{ -ik_z Y_n(\alpha_2 R_1) \}$$

$$a_{10,10}^{(n)} = -\left\{ -\left[ \frac{n+1}{R_1} J_{n+1}(\beta_2 R_1) + \beta_2 J_{n+1}^{'}(\beta_2 R_1) \right] \right\}$$

$$a_{10,11}^{(n)} = -\left\{ -\left[ \frac{n+1}{R_1} Y_{n+1}(\beta_2 R_1) + \beta_2 Y_{n+1}^{'}(\beta_2 R_1) \right] \right\}$$

$$a_{11,8}^{(n)} = \rho_2 c_{s2}^2 \left\{ \frac{2n}{R_2^2} J_n(\alpha_2 R_2) - \frac{2n\alpha_2}{R_2} J_n^{'}(\alpha_2 R_2) \right\}$$

$$a_{11,9}^{(n)} = \rho_2 c_{s2}^2 \left\{ \frac{2n}{R_2^2} Y_n(\alpha_2 R_2) - \frac{2n\alpha_2}{R_2} Y_n^{'}(\alpha_2 R_2) \right\}$$

$$a_{11,10}^{(n)} = \rho_2 c_{s2}^2 \left\{ \frac{i(n+1)}{R_2} k_z J_{n+1}(\beta_2 R_2) - ik_z \beta_2 J_{n+1}^{'}(\beta_2 R_2) \right\}$$

$$a_{11,11}^{(n)} = \rho_2 c_{s2}^2 \left\{ \frac{i(n+1)}{R_2} k_z Y_{n+1}(\beta_2 R_2) - ik_z \beta_2 Y_{n+1}^{'}(\beta_2 R_2) \right\}$$

$$a_{11,12}^{(n)} = \rho_2 c_{s2}^2 \left\{ -\left( \frac{n}{R_2} \right)^2 J_n(\beta_2 R_2) + \frac{\beta_2}{R_2} J_n^{'}(\beta_2 R_2) - \beta_2^2 J_n^{''}(\beta_2 R_2) \right\}$$

$$a_{11,13}^{(n)} = \rho_2 c_{s2}^2 \left\{ -\left(\frac{n}{R_2}\right)^2 Y_n(\beta_2 R_2) + \frac{\beta_2}{R_2} Y_n^{'}(\beta_2 R_2) - \beta_2^2 Y_n^{''}(\beta_2 R_2) \right\}$$

$$a_{12,8}^{(n)} = \rho_2 c_{s2}^2 \left\{ -2ik_z \alpha_2 J_n^{'}(\alpha_2 R_2) \right\}$$

$$a_{12,9}^{(n)} = \rho_2 c_{s2}^2 \left\{ -2ik_z \alpha_2 Y_n^{'}(\alpha_2 R_2) \right\}$$

$$a_{12,10}^{(n)} = \rho_2 c_{s2}^2 \left\{ \left( \frac{n+1}{R_2^2} - k_z^2 \right) J_{n+1}(\beta_2 R_2) \right.$$

$$\left. - \frac{n+1}{R_2} \beta_2 J_{n+1}^{'}(\beta_2 R_2) - \beta_2^2 J_{n+1}^{''}(\beta_2 R_2) \right\}$$

$$a_{12,11}^{(n)} = \rho_2 c_{s2}^2 \left\{ \left( \frac{n+1}{R_2^2} - k_z^2 \right) Y_{n+1}(\beta_2 R_2) \right.$$

$$\left. - \frac{n+1}{R_2} \beta_2 Y_{n+1}^{'}(\beta_2 R_2) - \beta_2^2 Y_{n+1}^{''}(\beta_2 R_2) \right\}$$

$$a_{12,12}^{(n)} = \rho_2 c_{s2}^2 \left\{ -\frac{ink_z}{R_2} J_n(\beta_2 R_2) \right\}$$

$$a_{12,13}^{(n)} = \rho_2 c_{s2}^2 \left\{ -\frac{ink_z}{R_2} Y_n(\beta_2 R_2) \right\}$$

$$a_{13,8}^{(n)} = \rho_3 \omega^2 \alpha_2 J_n^{'}(\alpha_2 R_2)$$

$$a_{13,9}^{(n)} = \rho_3 \omega^2 \alpha_2 Y_n^{'}(\alpha_2 R_2)$$

$$a_{13,10}^{(n)} = -i\rho_3\omega^2 k_z J_{n+1}(\beta_2 R_2)$$

$$a_{13,11}^{(n)} = -i\rho_3\omega^2 k_z Y_{n+1}(\beta_2 R_2)$$

$$a_{13,12}^{(n)} = \rho_3\omega^2 \frac{n}{R_2} J_n(\beta_2 R_2)$$

$$a_{13,13}^{(n)} = \rho_3\omega^2 \frac{n}{R_2} Y_n(\beta_2 R_2)$$

$$a_{13,14}^{(n)} = -k_3 \cos\varphi_3 \, J_n^{'}\left(k_3 R_2 \cos\varphi_3\right)$$

$$a_{14,8}^{(n)} = \rho_2 c_{d2}^2 \alpha_2^2 J_n^{''}(\alpha_2 R_2)$$
$$+\rho_2\left(c_{d2}^2 - 2c_{s2}^2\right)\left[-\left(\left(\frac{n}{R_2}\right)^2 + k_z^2\right)J_n(\alpha_2 R_2) + \frac{\alpha_2}{R_2} J_n^{'}(\alpha_2 R_2)\right]$$

$$a_{14,9}^{(n)} = \rho_2 c_{d2}^2 \alpha_2^2 Y_n^{''}(\alpha_2 R_2)$$
$$+\rho_2\left(c_{d2}^2 - 2c_{s2}^2\right)\left[-\left(\left(\frac{n}{R_2}\right)^2 + k_z^2\right)Y_n(\alpha_2 R_2) + \frac{\alpha_2}{R_2} Y_n^{'}(\alpha_2 R_2)\right]$$

$$a_{14,10}^{(n)} = \rho_2 c_{d2}^2\left[-ik_z\beta_2 J_{n+1}^{'}\left(\beta_2 R_2\right)\right]$$
$$+\rho_2\left(c_{d2}^2 - 2c_{s2}^2\right)\left[ik_z\beta_2 J_{n+1}^{'}\left(\beta_2 R_2\right)\right]$$

$$a_{14,11}^{(n)} = \rho_2 c_{d2}^2\left[-ik_z\beta_2 Y_{n+1}^{'}\left(\beta_2 R_2\right)\right]$$
$$+\rho_2\left(c_{d2}^2 - 2c_{s2}^2\right)\left[ik_z\beta_2 Y_{n+1}^{'}\left(\beta_2 R_2\right)\right.$$

$$a_{14,12}^{(n)} = \rho_2 c_{d2}^2 \left[ -\frac{n}{R_2^2} J_n(\beta_2 R_2) + \frac{n}{R_2} \beta_2 J_n^{'}(\beta_2 R_2) \right]$$

$$+\rho_2 \left( c_{d2}^2 - 2c_{s2}^2 \right) \left[ -\frac{n}{R_2} \beta_2 J_n^{'}(\beta_2 R_2) + \frac{n}{R_2^2} J_n(\beta_2 R_2) \right]$$

$$a_{14,13}^{(n)} = \rho_2 c_{d2}^2 \left[ -\frac{n}{R_2^2} Y_n(\beta_2 R_2) + \frac{n}{R_2} \beta_2 Y_n^{'}(\beta_2 R_2) \right]$$

$$+\rho_2 \left( c_{d2}^2 - 2c_{s2}^2 \right) \left[ -\frac{n}{R_2} \beta_2 Y_n^{'}(\beta_2 R_2) + \frac{n}{R_2^2} Y_n(\beta_2 R_2) \right]$$

$$a_{14,14}^{(n)} = J_n \left( k_3 R_2 \cos\varphi_3 \right)$$

$$b_1^{(n)} = -J_n \left( k_0 R_0 \cos\varphi \right)$$

$$b_2^{(n)} = k_0 \cos\varphi J_n^{'} \left( k_0 R_0 \cos\varphi \right)$$

# APPENDIX B
# SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

$$
\begin{bmatrix}
\alpha_{11}^{(n)} & \alpha_{12}^{(n)} & \alpha_{13}^{(n)} & \alpha_{14}^{(n)} & \alpha_{15}^{(n)} & \alpha_{16}^{(n)} & \alpha_{17}^{(n)} & & & & & & & \\
\alpha_{21}^{(n)} & \alpha_{22}^{(n)} & \alpha_{23}^{(n)} & \alpha_{24}^{(n)} & \alpha_{25}^{(n)} & \alpha_{26}^{(n)} & \alpha_{27}^{(n)} & & & & & & & \\
 & \alpha_{32}^{(n)} & \alpha_{33}^{(n)} & \alpha_{34}^{(n)} & \alpha_{35}^{(n)} & \alpha_{36}^{(n)} & \alpha_{37}^{(n)} & & & & & & & \\
 & \alpha_{42}^{(n)} & \alpha_{43}^{(n)} & \alpha_{44}^{(n)} & \alpha_{45}^{(n)} & \alpha_{46}^{(n)} & \alpha_{47}^{(n)} & & & & & & & \\
 & \alpha_{52}^{(n)} & \alpha_{53}^{(n)} & \alpha_{54}^{(n)} & \alpha_{55}^{(n)} & \alpha_{56}^{(n)} & \alpha_{57}^{(n)} & \alpha_{58}^{(n)} & \alpha_{59}^{(n)} & \alpha_{5,10}^{(n)} & \alpha_{5,11}^{(n)} & \alpha_{5,12}^{(n)} & \alpha_{5,13}^{(n)} & \\
 & \alpha_{62}^{(n)} & \alpha_{63}^{(n)} & \alpha_{64}^{(n)} & \alpha_{65}^{(n)} & \alpha_{66}^{(n)} & \alpha_{67}^{(n)} & \alpha_{68}^{(n)} & \alpha_{69}^{(n)} & \alpha_{6,10}^{(n)} & \alpha_{6,11}^{(n)} & \alpha_{6,12}^{(n)} & \alpha_{6,13}^{(n)} & \\
 & \alpha_{72}^{(n)} & \alpha_{73}^{(n)} & \alpha_{74}^{(n)} & \alpha_{75}^{(n)} & \alpha_{76}^{(n)} & \alpha_{77}^{(n)} & \alpha_{78}^{(n)} & \alpha_{79}^{(n)} & \alpha_{7,10}^{(n)} & \alpha_{7,11}^{(n)} & \alpha_{7,12}^{(n)} & \alpha_{7,13}^{(n)} & \\
 & \alpha_{82}^{(n)} & \alpha_{83}^{(n)} & \alpha_{84}^{(n)} & \alpha_{85}^{(n)} & \alpha_{86}^{(n)} & \alpha_{87}^{(n)} & \alpha_{88}^{(n)} & \alpha_{89}^{(n)} & \alpha_{8,10}^{(n)} & \alpha_{8,11}^{(n)} & \alpha_{8,12}^{(n)} & \alpha_{8,13}^{(n)} & \\
 & \alpha_{92}^{(n)} & \alpha_{93}^{(n)} & \alpha_{94}^{(n)} & \alpha_{95}^{(n)} & \alpha_{96}^{(n)} & \alpha_{97}^{(n)} & \alpha_{98}^{(n)} & \alpha_{99}^{(n)} & \alpha_{9,10}^{(n)} & \alpha_{9,11}^{(n)} & \alpha_{9,12}^{(n)} & \alpha_{9,13}^{(n)} & \\
 & \alpha_{10,2}^{(n)} & \alpha_{10,3}^{(n)} & \alpha_{10,4}^{(n)} & \alpha_{10,5}^{(n)} & 0 & 0 & \alpha_{10,8}^{(n)} & \alpha_{10,9}^{(n)} & \alpha_{10,10}^{(n)} & \alpha_{10,11}^{(n)} & 0 & 0 & \\
 & & & & & & & \alpha_{11,8}^{(n)} & \alpha_{11,9}^{(n)} & \alpha_{11,10}^{(n)} & \alpha_{11,11}^{(n)} & \alpha_{11,12}^{(n)} & \alpha_{11,13}^{(n)} & \\
 & & & & & & & \alpha_{12,8}^{(n)} & \alpha_{12,9}^{(n)} & \alpha_{12,10}^{(n)} & \alpha_{12,11}^{(n)} & \alpha_{12,12}^{(n)} & \alpha_{12,13}^{(n)} & \\
 & & & & & & & \alpha_{13,8}^{(n)} & \alpha_{13,9}^{(n)} & \alpha_{13,10}^{(n)} & \alpha_{13,11}^{(n)} & \alpha_{13,12}^{(n)} & \alpha_{13,13}^{(n)} & \alpha_{13,14}^{(n)} \\
 & & & & & & & \alpha_{14,8}^{(n)} & \alpha_{14,9}^{(n)} & \alpha_{14,10}^{(n)} & \alpha_{14,11}^{(n)} & \alpha_{14,12}^{(n)} & \alpha_{14,13}^{(n)} & \alpha_{14,14}^{(n)}
\end{bmatrix}
\begin{bmatrix}
A_0^{(n)} \\ A_{1,1}^{(n)} \\ B_{1,1}^{(n)} \\ A_{2,1}^{(n)} \\ B_{2,1}^{(n)} \\ A_{3,1}^{(n)} \\ B_{3,1}^{(n)} \\ A_{1,2}^{(n)} \\ B_{1,2}^{(n)} \\ A_{2,2}^{(n)} \\ B_{2,2}^{(n)} \\ A_{3,2}^{(n)} \\ B_{3,2}^{(n)} \\ B_0^{(n)}
\end{bmatrix}
=
\begin{bmatrix}
b_1^{(n)} \\ b_2^{(n)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

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